First steps in the formalization of convex polyhedra in Coq

"Solvers Principles and Architectures" Seminar, Rennes

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Formal proving = "checking" proof using computers
… by writing the proof in a **proof assistant**

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Recent achievements

- completion of the proof of Kepler conjecture (Hales et al., 2017)
- Feit–Thompson theorem (Gonthier et al., 2013)

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A **(convex) polyhedron** is the set of the solutions $x \in \mathbb{R}^n$ of a system of linear (affine) inequalities:

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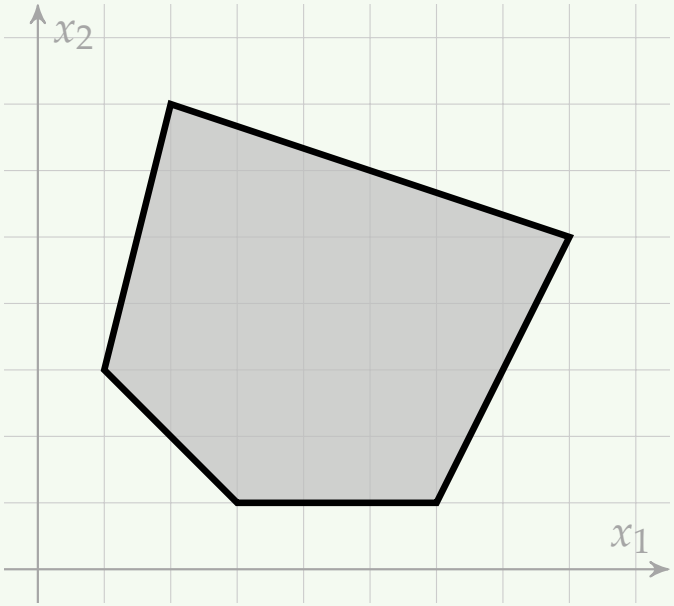

$$\begin{pmatrix} 1 & 1 \\ 4 & -1 \\ -1 & -3 \\ -2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 4 \\ 1 \\ -23 \\ -11 \end{pmatrix}$$

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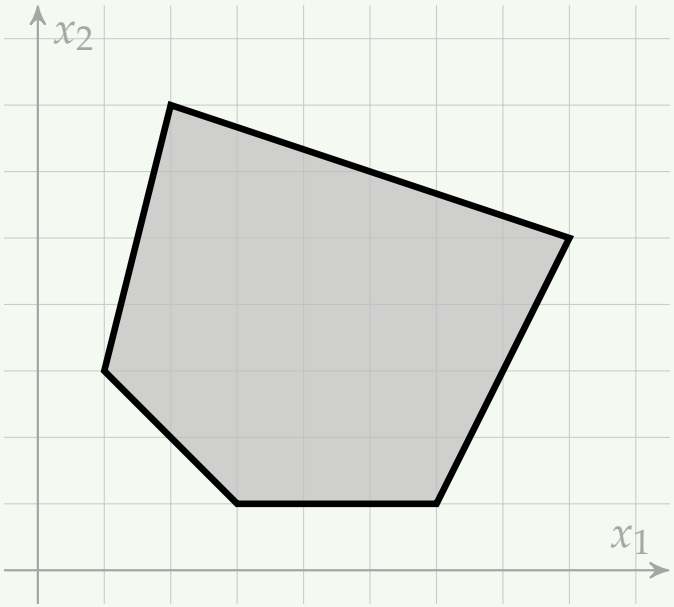

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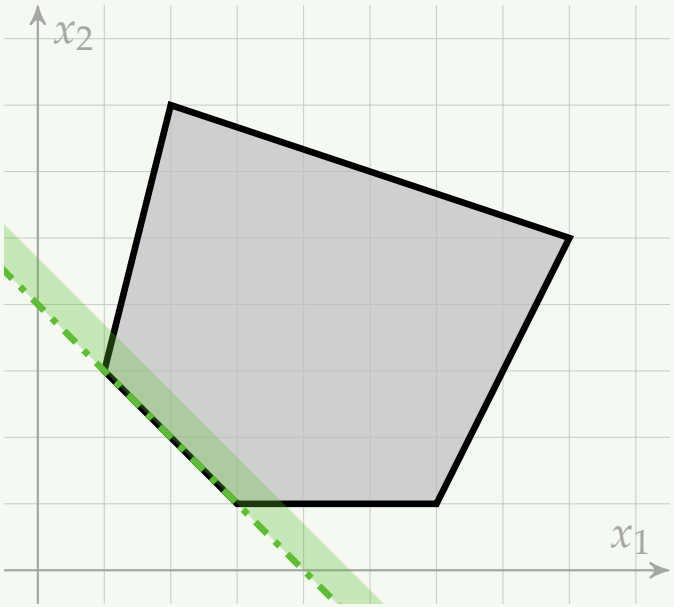

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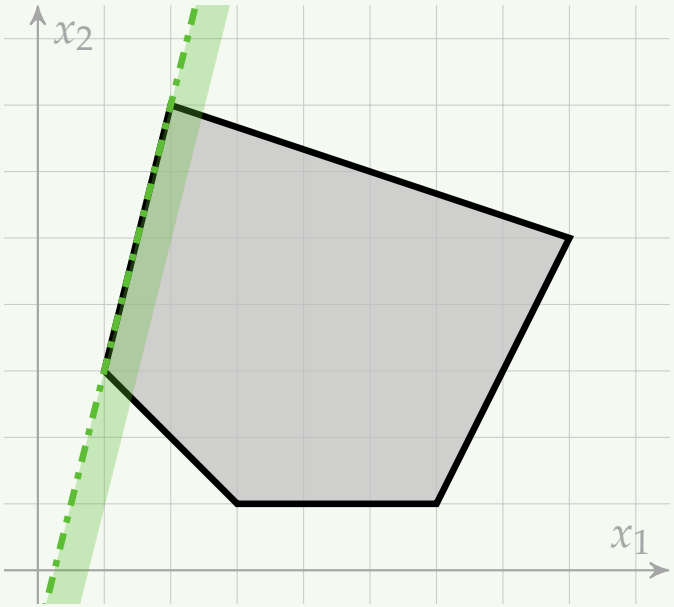

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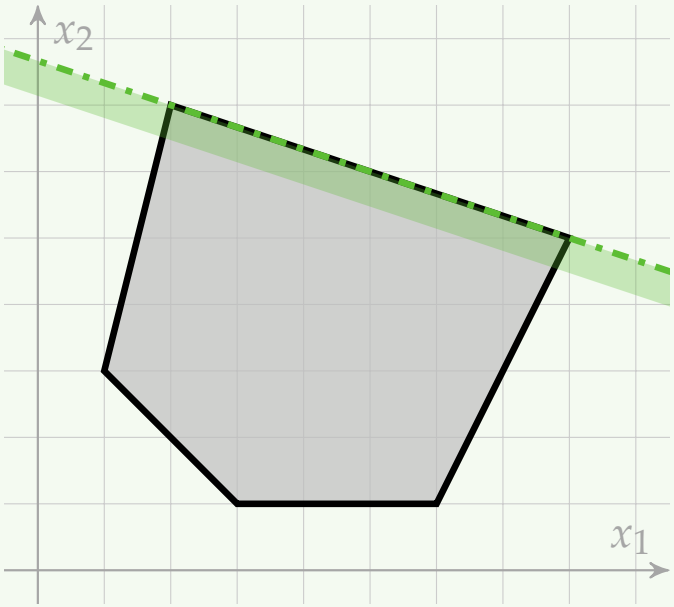

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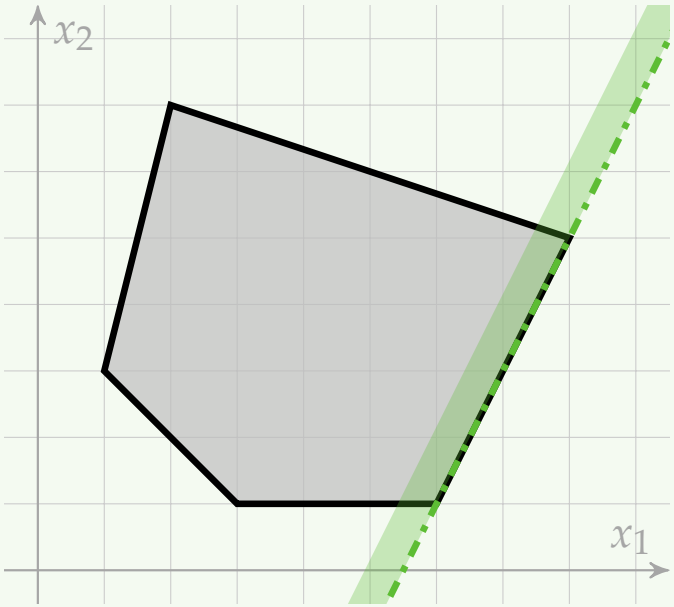

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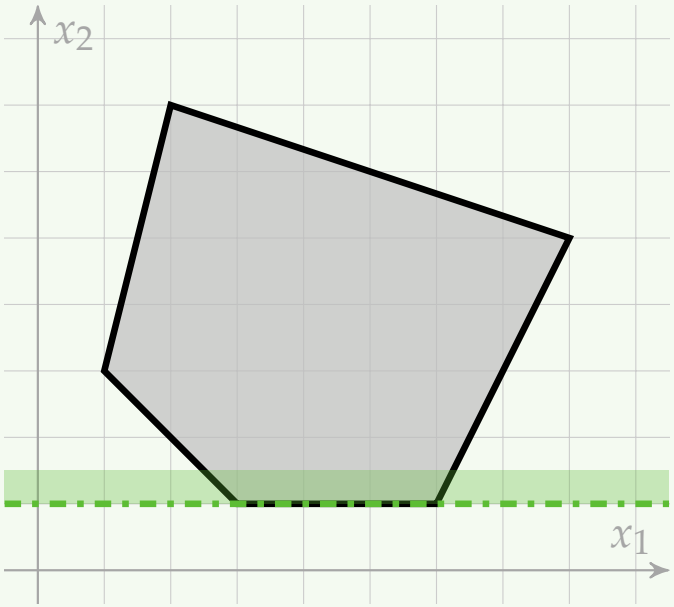

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Applied maths	optimization, operations research, control theory
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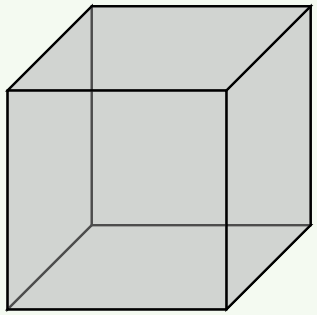
Convex polyhedra have a rich combinatorial flavor via their faces:

facets, …, edges, vertices

The faces of a cube



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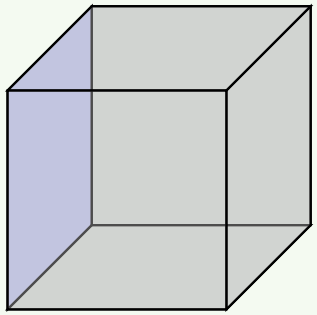

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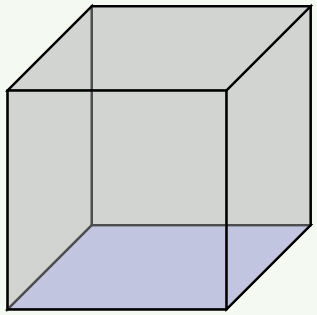

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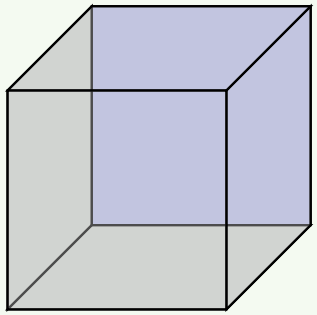

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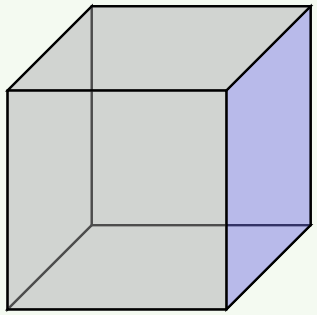

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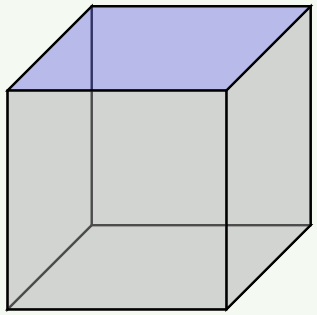

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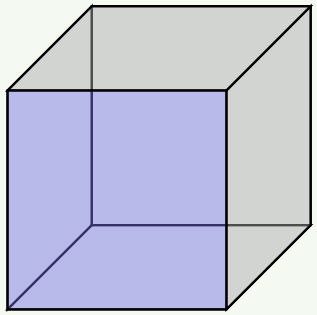

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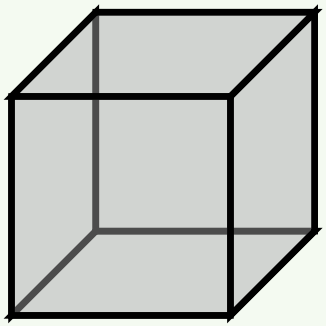

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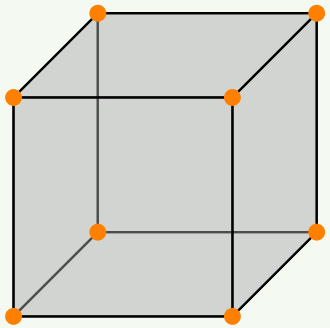

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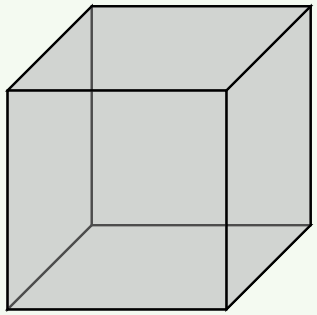

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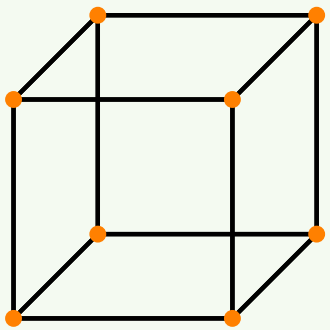

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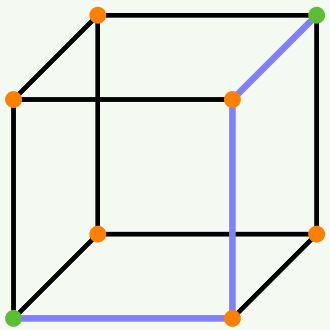

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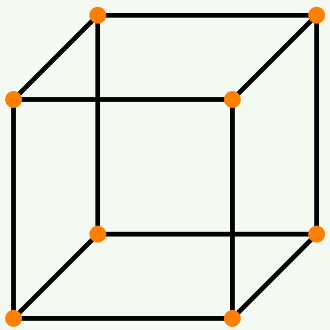

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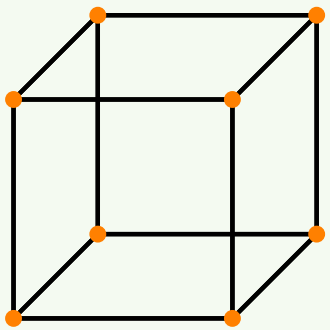

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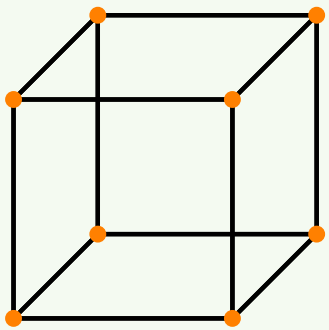

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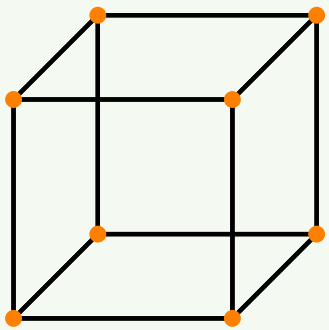

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Open problem

Is there a **polynomial upper bound** on the diameter?

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Linear programming

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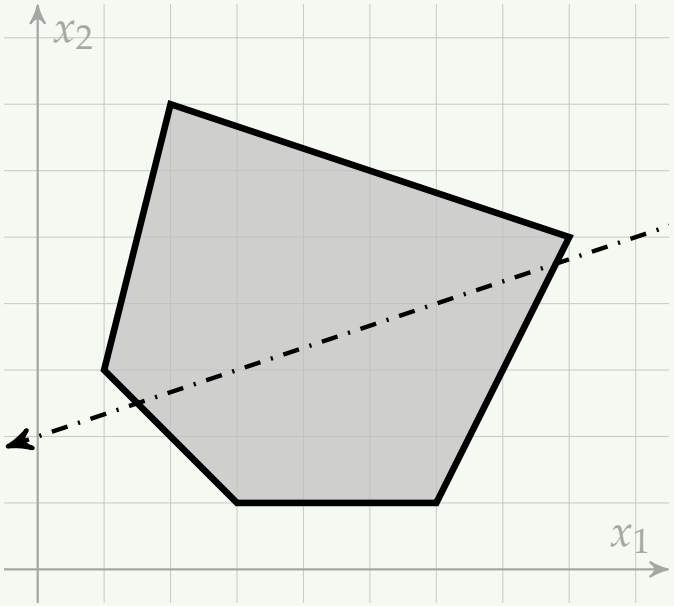

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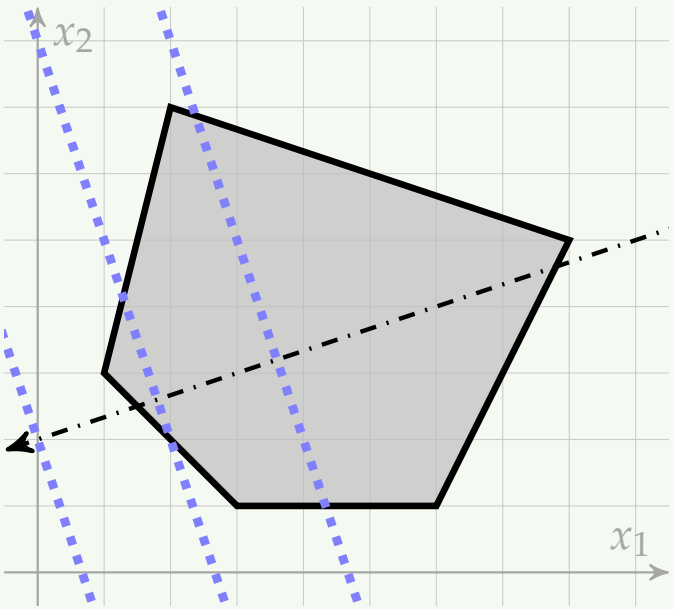

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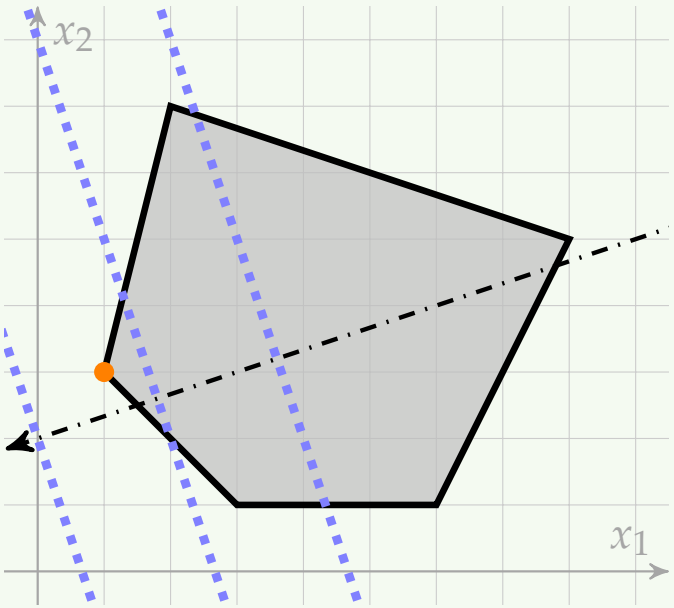

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- number of arithmetic operations bounded by a polynomial in the **dimension** of the problem, i.e. $\sim m \times n$.

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- provide **rigorous** proof of theorems relying on **informal computations**
  $\Longrightarrow$ "formal experimental maths"

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A computational motivation

Counterexample of Matschke, Santos, and Weibel (2015) to Hirsch conjecture:
a polytope in dimension 20, with 40 facets and… 36 425 vertices

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1. Coq is known for its **computational abilities**
   $+$ computing that $a = b$ in Coq is a proof of $a = b$.

2. Coq has the **Mathematical Components** library (Gonthier et al., 2016)

MathComp library

- **large hierarchy** of mathematical objets: linear algebra, finite groups, algebraic numbers, representation theory, etc
- introduce **proven methodologies** for formalization of maths in Coq

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- no excluded middle law $P \vee \neg P$

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Constructive logic

- no excluded middle law $P \vee \neg P$
- no double negation elimination $\neg\neg P \Rightarrow P$, no proof by contradiction
- to show $\exists x.\, P(x)$, you need to **construct** an x such that $P(x)$ holds.

This talk

First steps of the formalization of the theory of convex polyhedra, in Coq.

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Implementation

The development is gathered in the library **Coq-Polyhedra**:

`github.com/nhojem/Coq-Polyhedra`

Outline of the talk

1. The simplex method: base block of the theory of convex polyhedra
2. Formalization of the simplex method
3. Concluding remarks and perspectives

Outline of the talk

Overcoming the intuitionistic nature of Coq

Formulas of Coq live in the sort `Prop`.

	`Prop`
	`True, False`
conjunction	`A /\ B`
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Problems

- a polyhedron is empty, or non-empty, or what?
- the formula `~(empty A b)` does not provide a point `x` s.t. `A *m x >=m b`

Overcoming the intuitionistic nature of Coq (2)

Formulas of Coq live in the sort `Prop`.

+ Boolean algebra, i.e., the type `bool := true | false`

	`Prop`	`bool`
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$\Longrightarrow$ The Coq formula `P` can be manipulated like in classical logic! case analysis, etc

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Assume that $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
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Solution

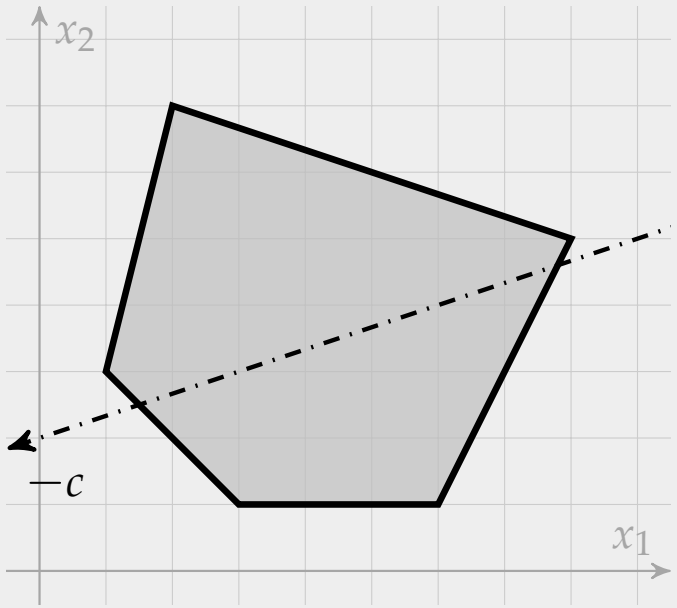
Implement **decision procedures** for emptiness, boundedness, membership, etc
$\Longrightarrow$ one building block: the simplex method in Coq

The simplex method

Linear programming

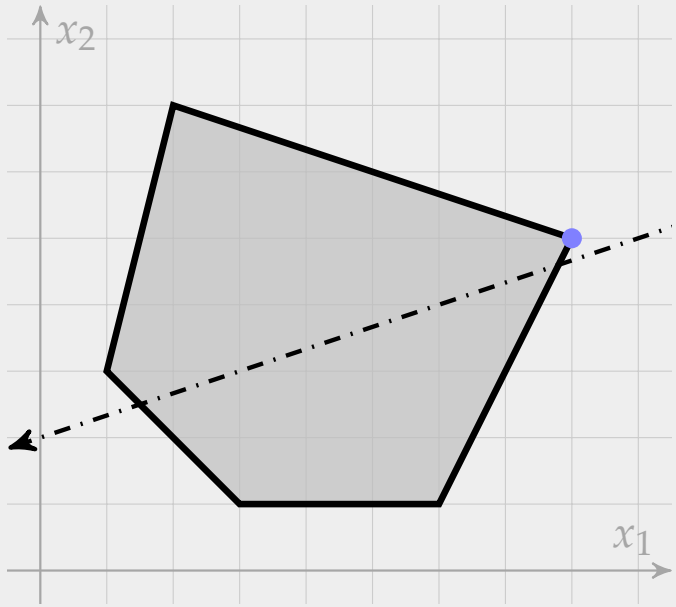
$$\begin{aligned} &\text{minimize} && \langle c, x \rangle \\ &\text{subject to} && Ax \geq b\,,\ x \in \mathbb{R}^n \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, and $\langle c, x \rangle := \sum_{i=1}^{n} c_i x_i$.



The simplex method (2)

1. starting from an initial vertex



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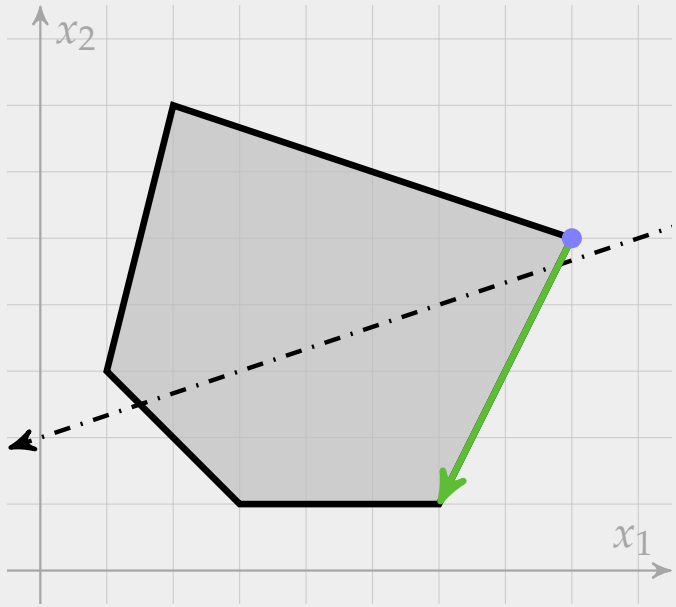
1. starting from an initial vertex
2. iterate over the vertex-edge graph while decreasing the objective function



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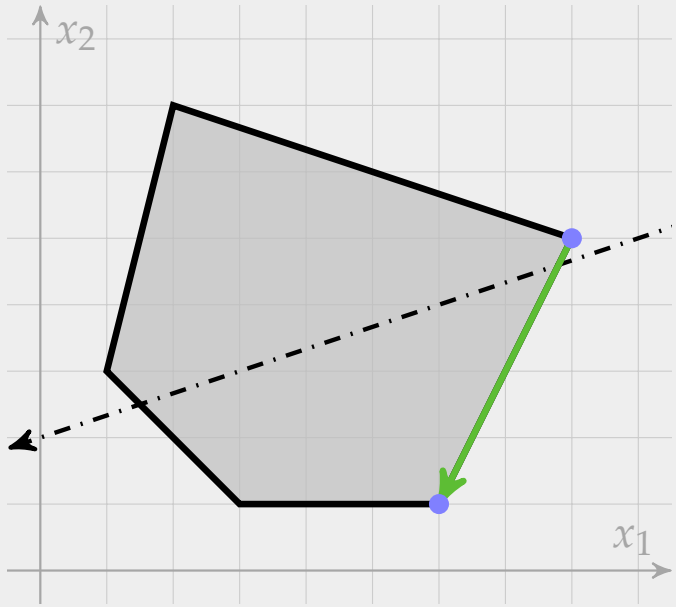

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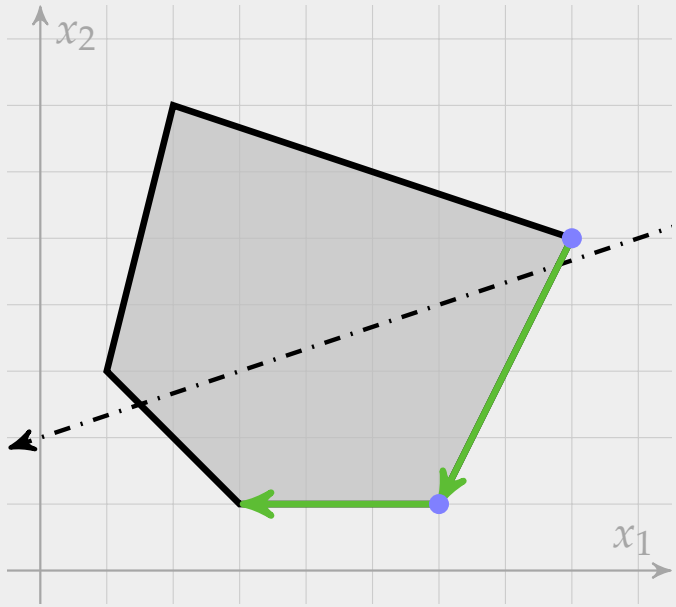

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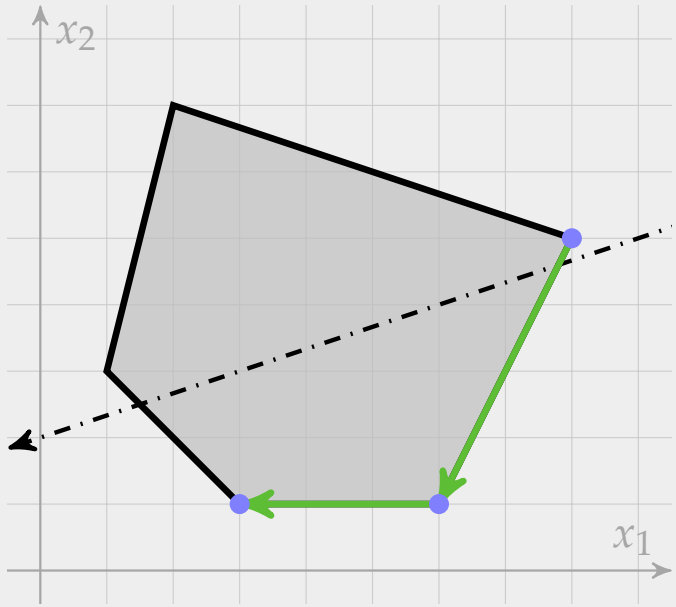

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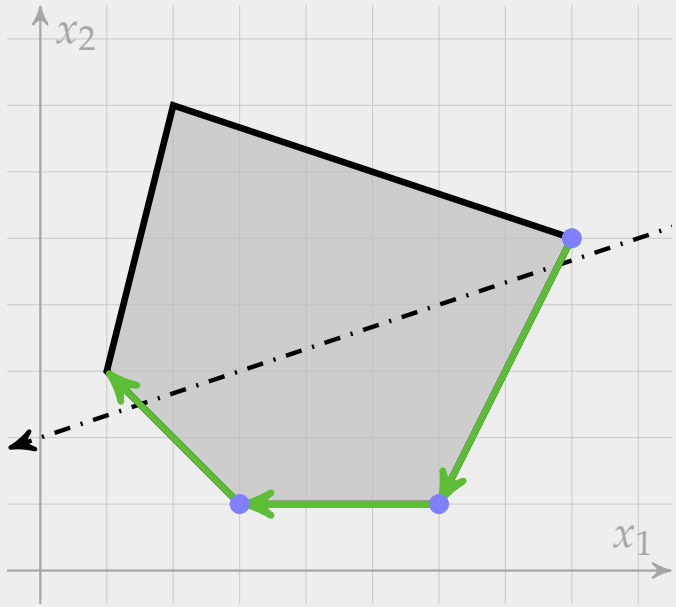

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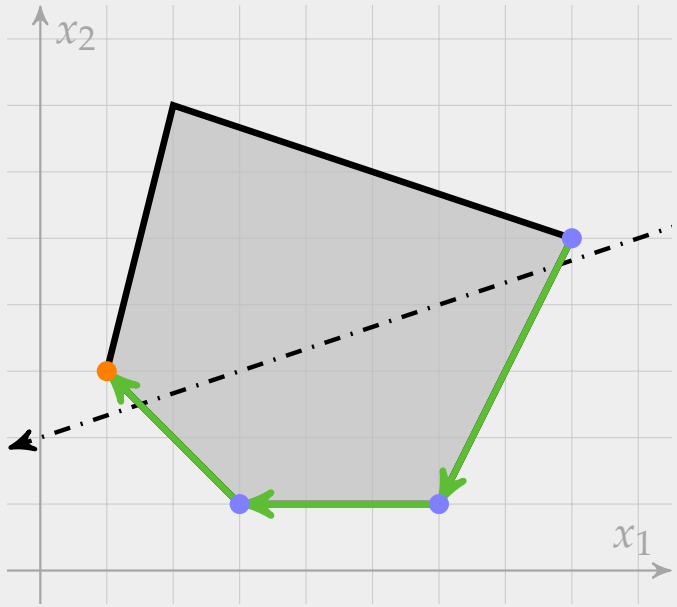
1. starting from an initial vertex
2. iterate over the vertex-edge graph while decreasing the objective function
3. up to finding an optimal vertex



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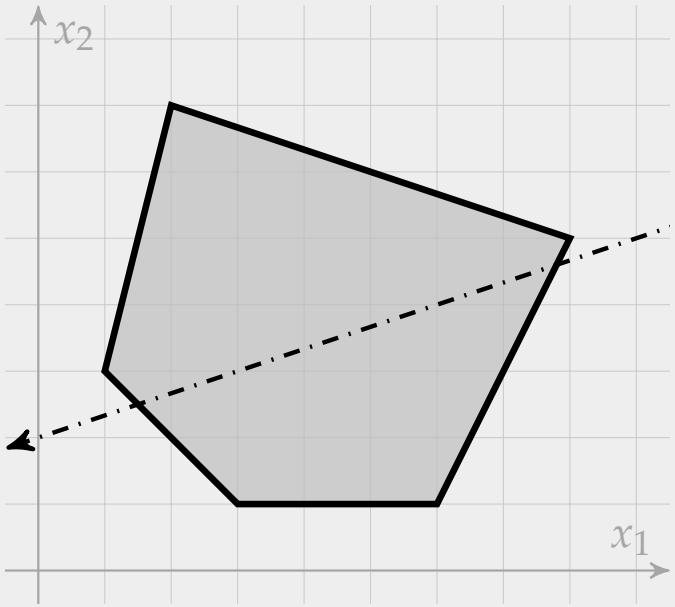

Three ingredients

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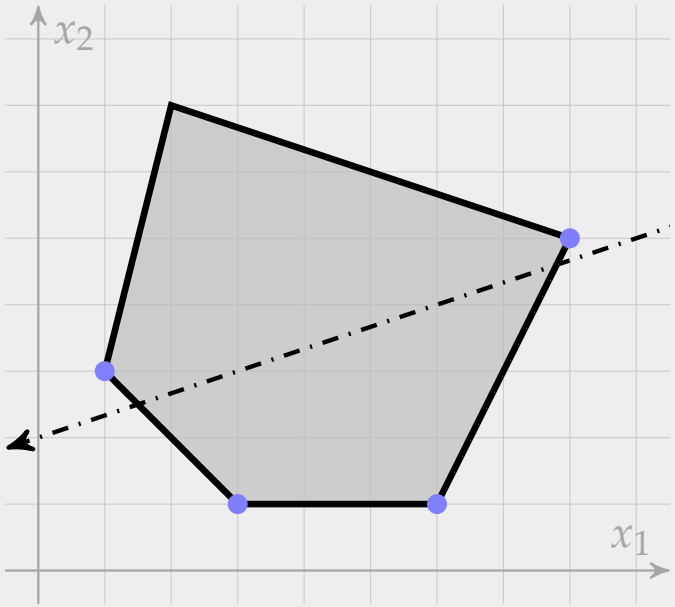

Three ingredients

- bases encode vertices

Definition

A **basis** is a subset $I \subset [m]$ of cardinality n such that the rows $(A_i)_{i \in I}$ are linearly independent.
The associated **basic point** is defined as:

$$x^I := A_I^{-1} b_I \,.$$

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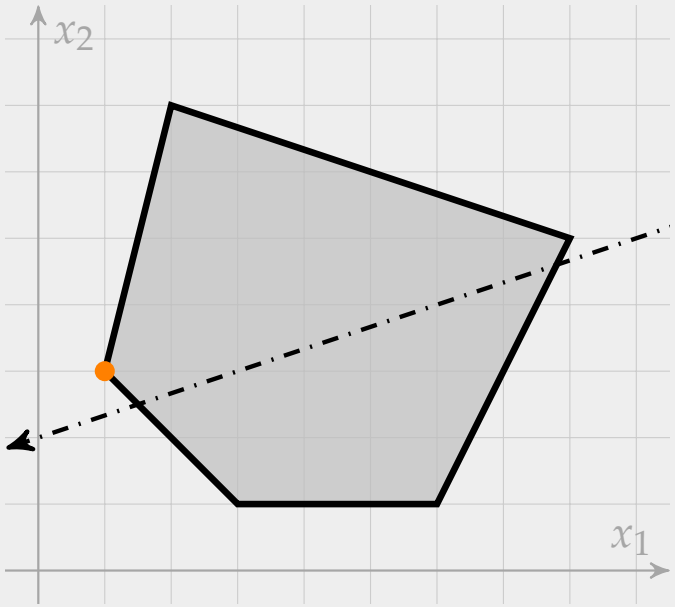

Three ingredients

- bases encode vertices
- reduced costs determine optimality

Reduced cost vector

The **reduced cost vector** at basis I is defined as $u^I := A_I^{-T} c_I$.
When **non-negative**, it provides a feasible element of the dual LP:

$$\text{maximize} \quad \langle b, u \rangle \quad \text{subject to} \quad A^T u = c\,,\ u \geq 0\,,\ u \in \mathbb{R}^m$$

which satisfies $\langle b, u^I \rangle = \langle c, x^I \rangle$.

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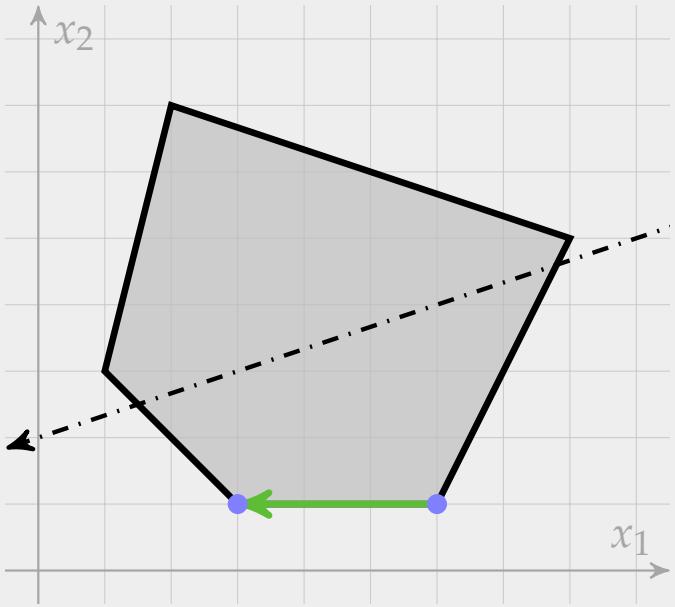

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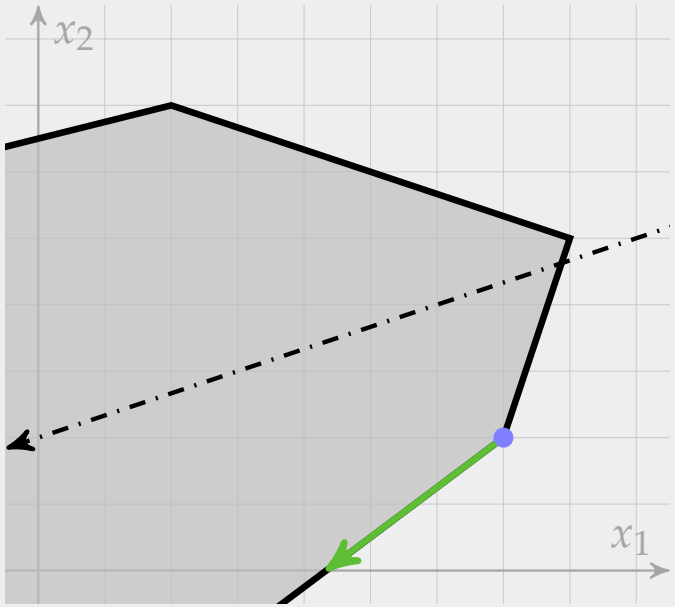

Three ingredients

- bases encode vertices
- reduced costs determine optimality
- pivoting switches from a vertex to another **or** determines if the LP is unbounded

Emptiness as a Boolean predicate

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Farkas Lemma

The polyhedron $\{x \in \mathbb{R}^n : Ax \geq b\}$ is empty if, and only if,

- there exists $d \in \mathbb{R}^m$ such that $A^T d = 0$ and $\langle b, d \rangle > 0$

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Suppose that we have formalized the simplex method in Coq:

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  match simplex [A^T -A^T 1:%M] 0 (-b) with
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  end.
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We get Farkas Lemma for free:

```
Theorem emptyP A b :
  reflect (exists d, A^T *m d = 0 /\ '[b,d] > 0) (empty A b).
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Follows from the correctness of the simplex method:

if `phase2` returns `Optimal_basis I`, then `reduced_cost I >=m 0`

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$$\equiv \quad \text{minimize} \quad 0 \quad \text{subject to} \quad Ax \geq b\,,\ x \in \mathbb{R}^n$$

```
Qed.
```

The simplex method is a constructive Swiss knife

Primal LP

$$\begin{aligned} \text{minimize} \quad & \langle c, x \rangle \\ \text{subject to} \quad & Ax \geq b\,,\ x \in \mathbb{R}^n \end{aligned}$$

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Theorem (Strong duality)

If both LP are feasible, there exists optimal solutions x^* and u^* of the primal and dual LP such that:

$$\langle c, x^* \rangle = \langle b, u^* \rangle$$

The simplex method is a constructive Swiss knife

Primal LP		**Dual LP**	
minimize	$\langle c, x \rangle$	maximize	$\langle b, u \rangle$
subject to	$Ax \geq b\,,\ x \in \mathbb{R}^n$	subject to	$A^T u = c\,,\ u \geq 0\,,\ u \in \mathbb{R}^m$

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If both LP are feasible, there exists optimal solutions x^* and u^* of the primal and dual LP such that:

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Proof sketch

The simplex returns an optimal basis I:

- `point_of_basis I` provides an optimal point of the primal;
- `reduced_cost I >=m 0` provides an optimal point of the dual.

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Other properties can be defined as Boolean predicate thanks to the simplex method: boundedness, membership to the convex hull, …

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Definition

A point $x \in \mathbb{R}^n$ belongs to the convex hull of the set $V = \{v^1, \ldots, v^p\}$ if

$$\exists \lambda \in \mathbb{R}^p\,, \quad x = \sum_{i=1}^{p} \lambda_i v^i \quad \text{where } \lambda \geq 0\,,\ \sum_{i=1}^{p} \lambda_i = 1\,.$$

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$\Longrightarrow$ membership amounts to the non-emptiness of a polyhedron in $\lambda \in \mathbb{R}^p$ (parametrized by x and V)

```
Definition is_in_convex_hull (x:'cV_n) :=
  let Ax := [V -V e^T -e^T 1%:M] in
  let bx := [x -x 1 -1 0] in
  ~~ (empty Ax bx) : bool.
```

The simplex method is a constructive Swiss knife (2)

Other properties can be defined as Boolean predicate thanks to the simplex method: boundedness, **membership to the convex hull**, …

Definition

A point $x \in \mathbb{R}^n$ belongs to the convex hull of the set $V = \{v^1, \ldots, v^p\}$ if

$$\exists \lambda \in \mathbb{R}^p, \quad x = \sum_{i=1}^{p} \lambda_i v^i \quad \text{where } \lambda \geq 0, \ \sum_{i=1}^{p} \lambda_i = 1 .$$

$\Longrightarrow$ membership amounts to the non-emptiness of a polyhedron in $\lambda \in \mathbb{R}^p$ (parametrized by x and V)

```
Definition is_in_convex_hull (x:'cV_n) :=
  let Ax := [V -V e^T -e^T 1%:M] in
  let bx := [x -x 1 -1 0] in
  ~~ (empty Ax bx) : bool.

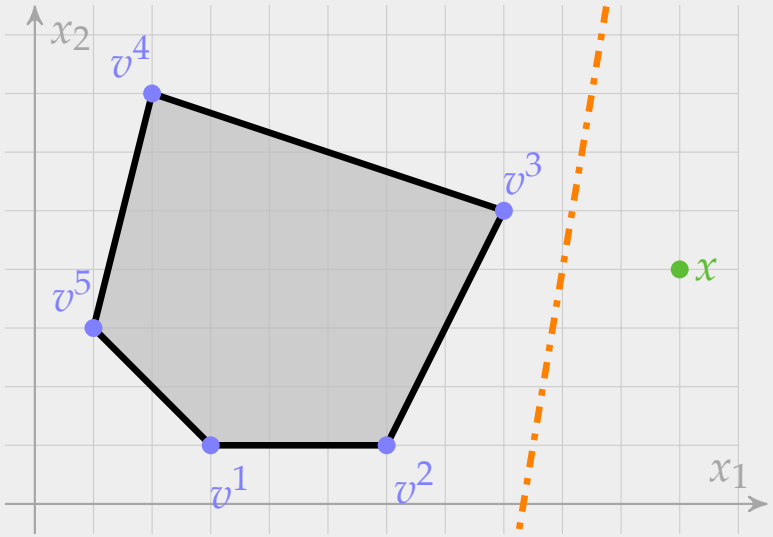
Lemma is_in_convex_hullP (x:'cV_n) :
  reflect
    (exists lambda,
      [/\ (lambda >=m 0), '[e, lambda] = 1 & x = V *m lambda])
    (is_in_convex_hull x).
```

The simplex method is a constructive Swiss knife (3)

Theorem (Separation result)

*The point x **does not** belong to the convex hull of V iff there exists $c \in \mathbb{R}^n$ such that*

$$\langle c, v^i \rangle > \langle c, x \rangle \ , \quad i = 1, \ldots, p$$



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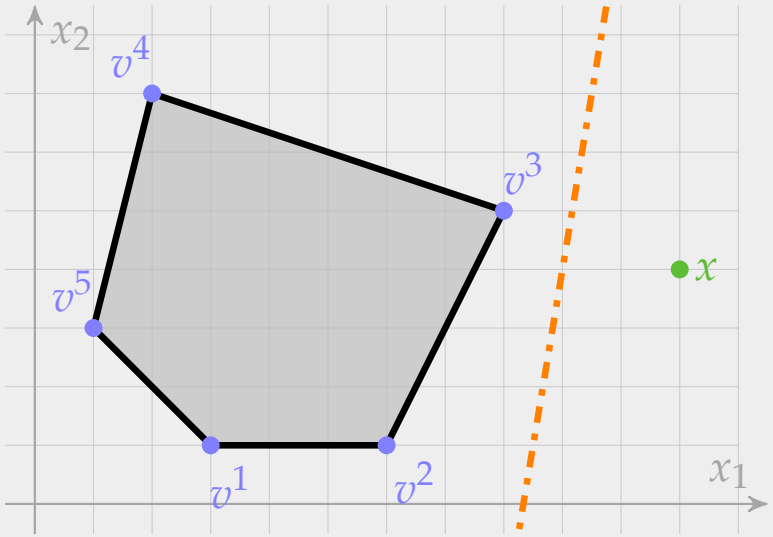


```
Theorem separation x :
  reflect (exists c, forall i, '[c, col i V] > '[c, x])
          ~~ (is_in_convex_hull x).
```

Proof sketch

The normal vector `c` is built from the (Farkas) emptiness certificate `d` of the polyhedron over $\lambda \in \mathbb{R}^p$.

The simplex method is a constructive Swiss knife (4)

Theorem (Minkowski)

Every bounded polyhedron is the convex hull of finitely many points.

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Theorem minkowski : bounded_polyhedron A b ->
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where:

- `=i` is the extensional equality
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Suppose that x lies in the polyhedron.

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$$\langle c, x \rangle < \langle c, z \rangle \quad \text{for all feasible basic point } z$$

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Suppose that x lies in the polyhedron.

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$$\langle c, z^* \rangle \leq \langle c, x \rangle < \langle c, z \rangle \quad \text{for all feasible basic point } z$$

where z^* is the basic point found by the simplex method.

`Qed.`

Outline of the talk

Bases

Definition

A **basis** is a subset $I \subset \{1, \ldots, m\}$ of cardinality n such that the system

$$A_i x = b_i\,, \quad i \in I$$

has a unique solution, called the **basic point**.

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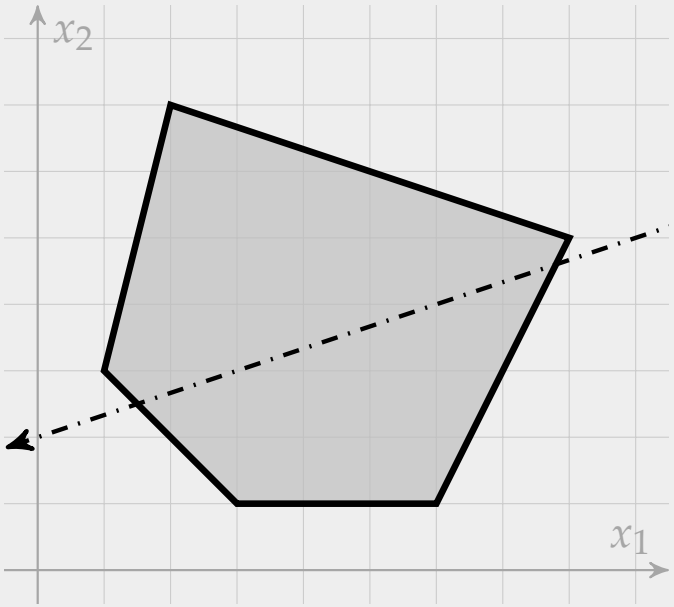

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\text{minimize} \quad & 3x_1 + x_2 \\
\text{subject to} \quad & x_1 + x_2 \geq 4 && \text{①} \\
& -x_1 - 3x_2 \geq -23 && \text{②} \\
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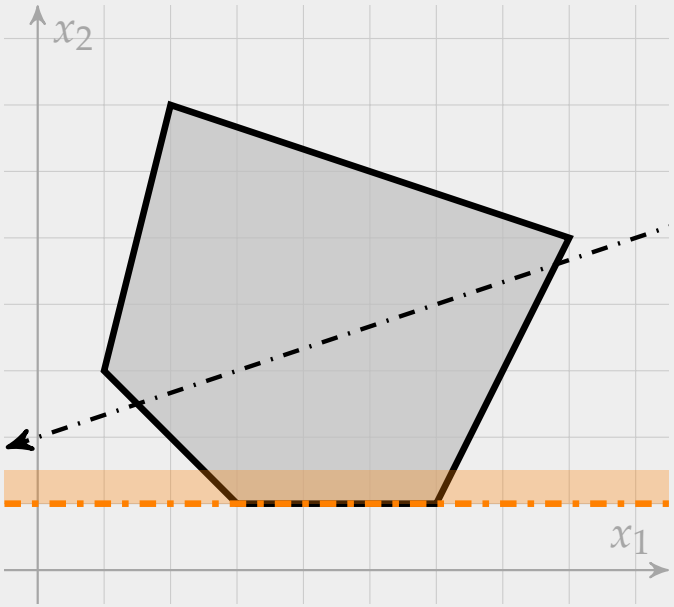

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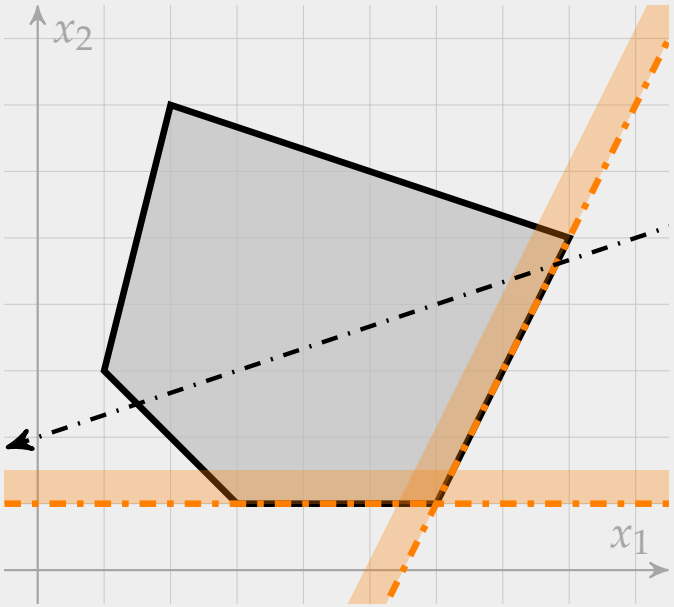

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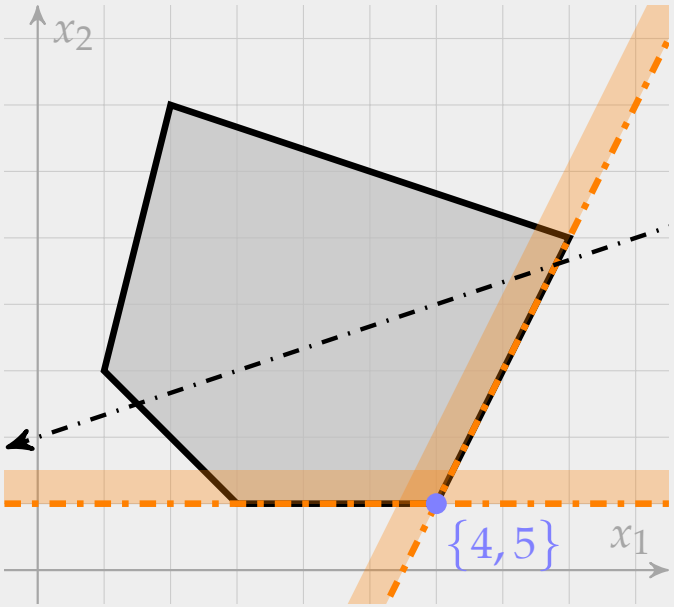

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= submatrix A_I

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Inductive feasible_basis :=
  FeasibleBasis (I:basis) of point_of_basis I \in polyhedron A b
```

where we have defined:

```
Definition point_of_basis (I:basis) :=
  (qinvmx [...] (row_submx A I)) *m (row_submx b I).
```

Reduced costs: optimality certificate

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Variable I : feasible_basis.
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$$\langle c, x\rangle = \langle u, A_I x\rangle = \langle \underbrace{u}_{\geq 0}, \underbrace{A_I x - b_I}_{\geq 0}\rangle + \langle u, b_I\rangle$$

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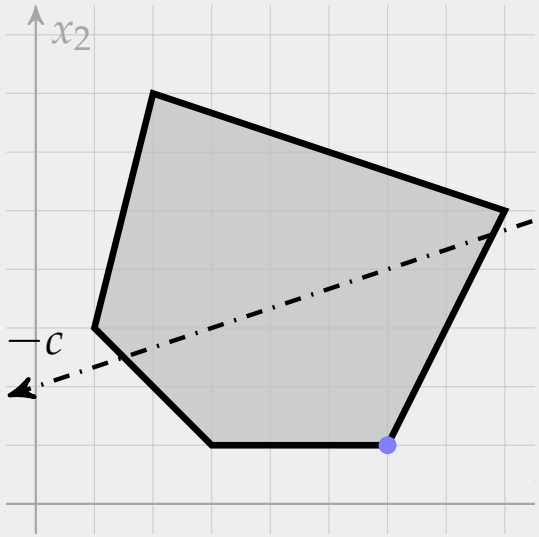
```
Qed.
```

Pivoting

If the reduced cost vector has some negative entries:

```
Variable i : 'I_#|I|.
Hypothesis [...] : reduced_cost i 0 < 0.
```



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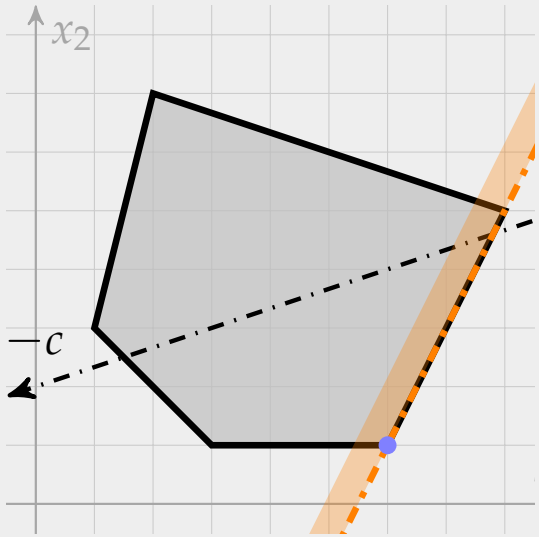

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we can build a direction vector which

- follows an incident edge
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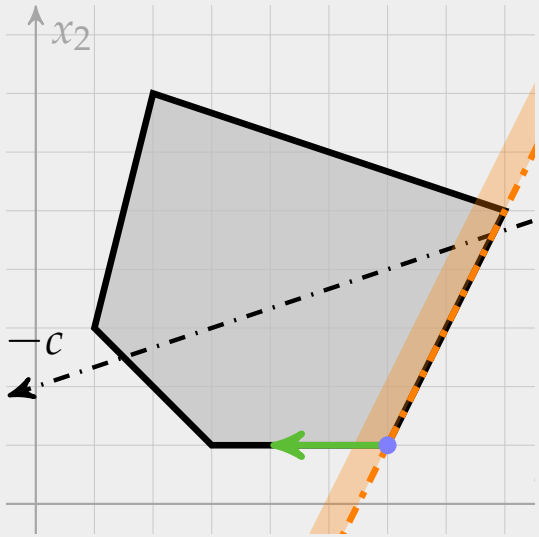

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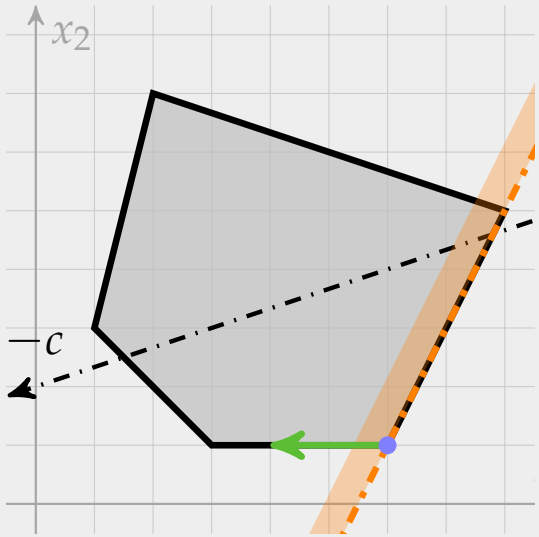


```
Definition direction :=
  let: ei := (delta_mx i 0) in
  (qinvmx [...] (row_submx A I)) *m ei.

Lemma direction_improvement :
  '[c, direction] < 0.
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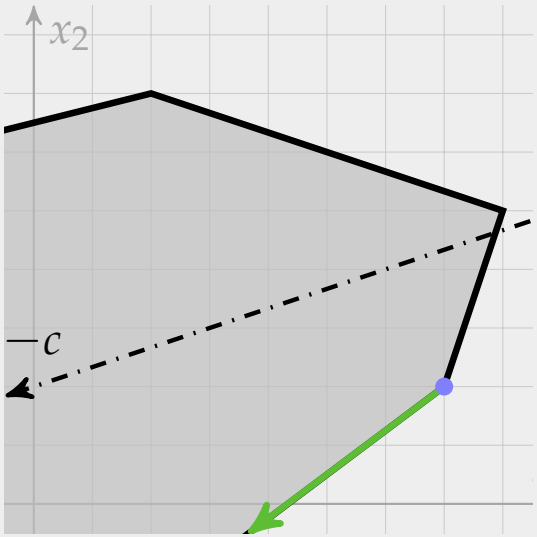

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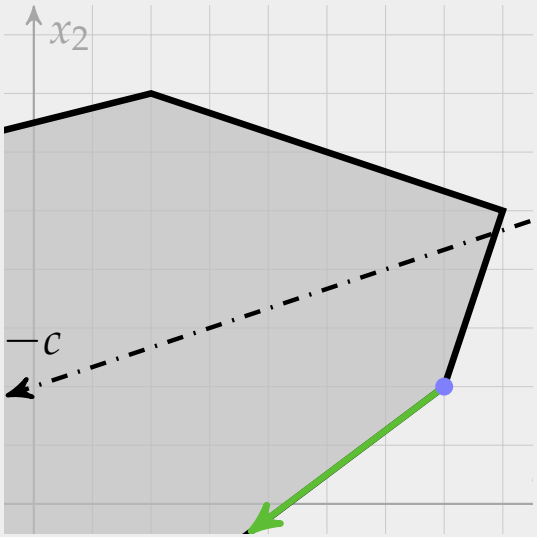

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Lemma unbounded_certificate : (A *m direction) >=m 0 ->
  forall M, exists x, (x \in polyhedron A b) /\ ('[c,x] < M)
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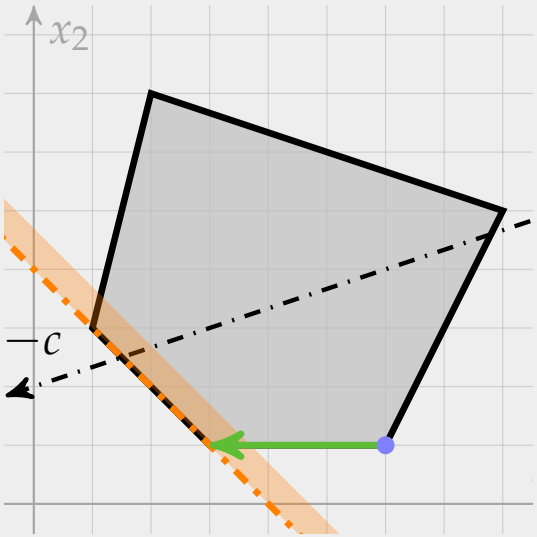

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Definition new_halfspace := [...]
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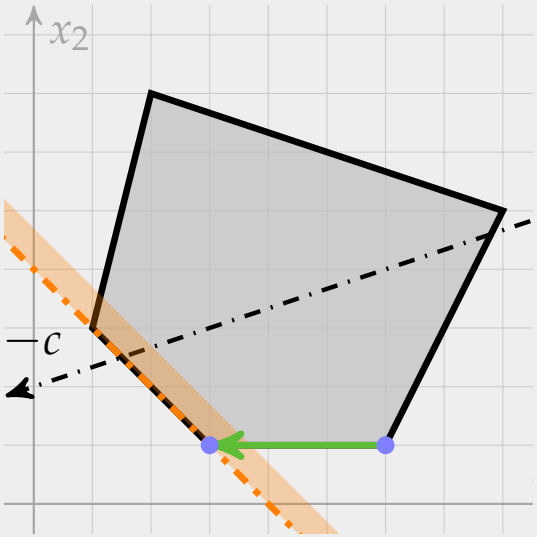

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Definition next_I :=
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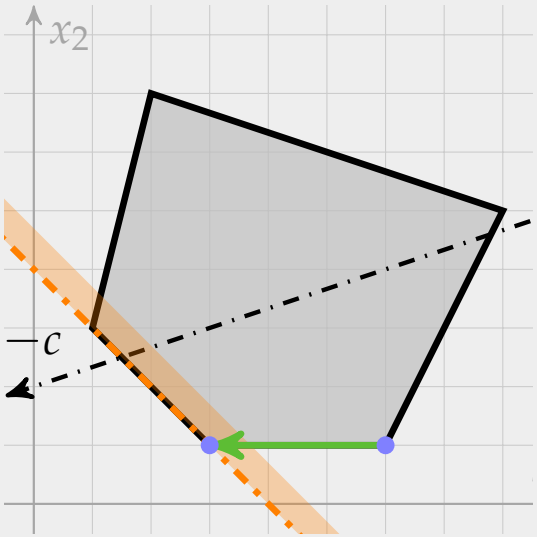

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$=$ removes `i` from `I`

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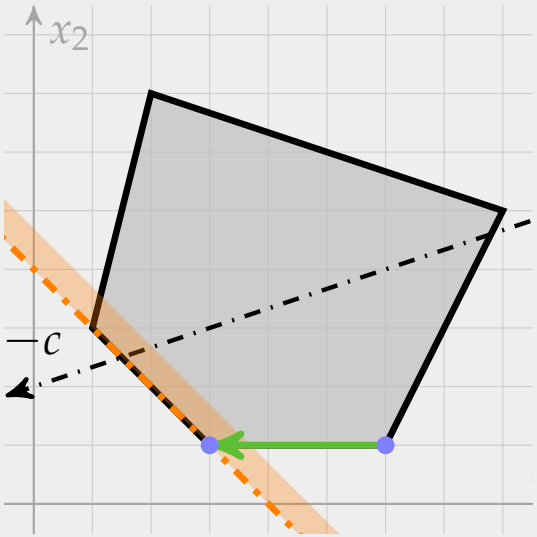

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= adds `new_halfspace`

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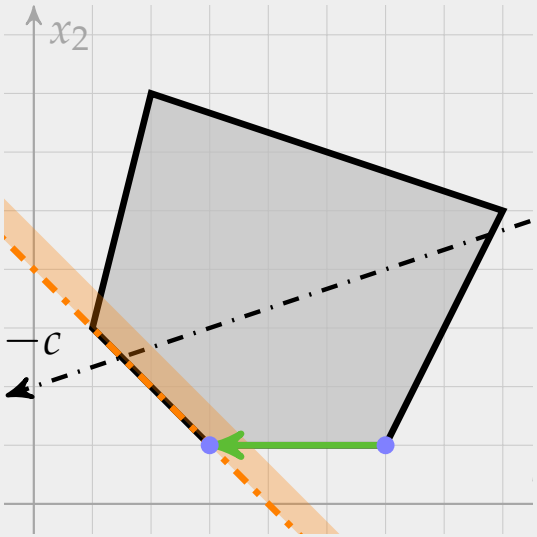

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Dealing with termination

The simplex methods iterates over the basic points with decreasing value:

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```

If at every step the inequality is **strict**, termination follows from the fact that there are finitely many bases:

```
Definition basis_height I :=
  #|[ set J: feasible_bases |
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The simplex methods iterates over the basic points with decreasing value:

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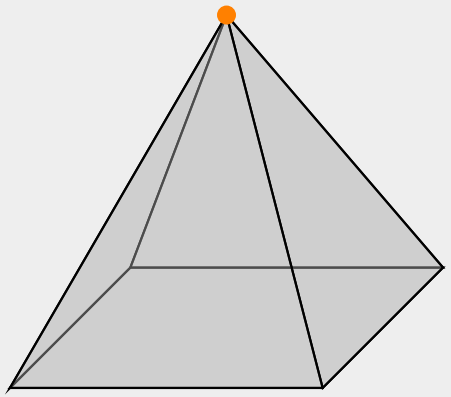

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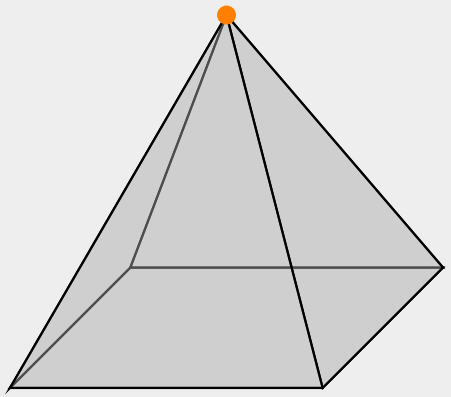

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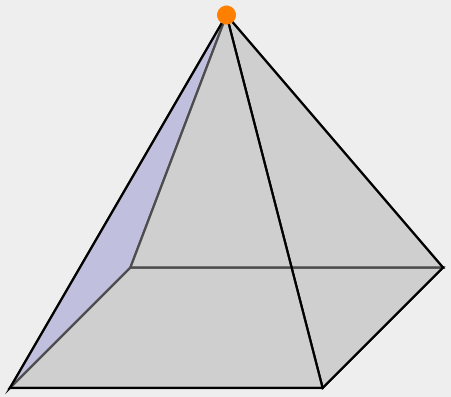

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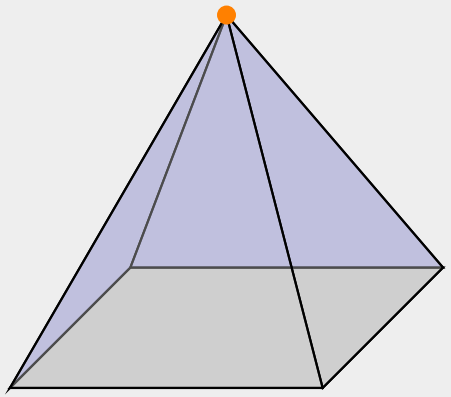

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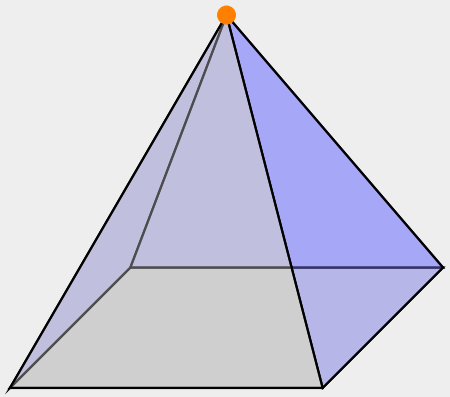

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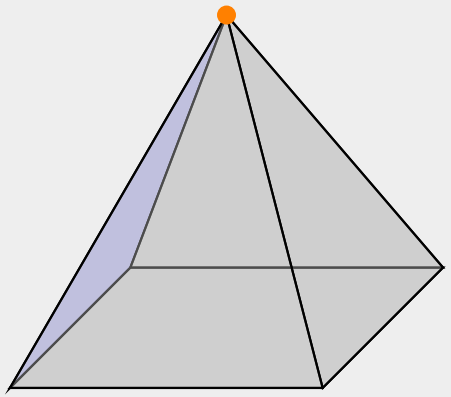

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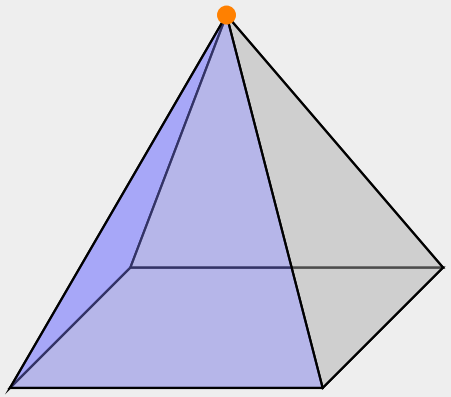

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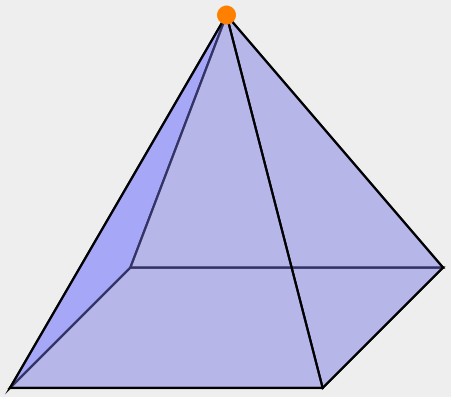

Dantzig's lexicographic rule

Degenerate bases disappear if we consider a slightly **perturbed** LP:

$$\text{minimize} \quad \langle c, x \rangle \quad \text{subject to} \quad Ax \geq \tilde{b}\,, \quad \text{with} \quad \tilde{b}_i := b_i - \varepsilon^i$$

where $0 < \varepsilon \ll 1$.

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Symbolic perturbation scheme

- every real v is now a polynomial in ε of degree $\leq m$

  $$v + v_1\varepsilon + v_2\varepsilon^2 + \cdots + v_m\varepsilon^m$$

  encoded as a row vector $(v, v_1, \ldots, v_m)$.

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- the usual order is replaced by the lexicographic order.

$$(v, v_1, \ldots, v_m) \leq_{\text{lex}} (w, w_1, \ldots, w_m)$$
$$\iff v + v_1\varepsilon + v_2\varepsilon^2 + \cdots + v_m\varepsilon^m \leq w + w_1\varepsilon + w_2\varepsilon^2 + \cdots + w_m\varepsilon^m$$
$$\text{for all } 0 < \varepsilon \ll 1\,.$$

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Observation

The perturbed polyhedron $\{x \in \mathbb{R}^n \mid Ax \geq \tilde{b}\}$ can be encoded by the set

$$\left\{x \in \mathbb{R}^{n\times(1+m)} \mid Ax \geq_{\text{lex}} \begin{pmatrix} b & -I_n \end{pmatrix}\right\}$$

Dantzig's lexicographic rule (2)

Remark: A and c are not perturbed

$\Longrightarrow$ bases and reduced cost vectors are left unchanged.

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Definition

A basis I is said to be **lex-feasible** if the perturbed basic point

$$\tilde{x}^I := (A_I)^{-1}\tilde{b}_I$$

belongs to the perturbed polyhedron.

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$$\tilde{x}^I := (A_I)^{-1} \begin{pmatrix} b_I & (-\delta_{ij})_{(i,j)\in I\times[m]} \end{pmatrix}$$

satisfies $A\tilde{x}^I \geq_{\text{lex}} \begin{pmatrix} b & -I_n \end{pmatrix}$.

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Proposition

Lex-feasible bases form a subset of feasible bases.

Proof sketch

The point $\tilde{x}^I$ is a perturbation of the basic point x^I:

```
Lemma [...] : point_of_basis I = col 0 (point_of_basis_pert I).
+ forall u v, u <=lex v -> u 0 0 <= v 0 0.
Qed.
```

Dantzig's lexicographic rule (3)

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Lemma no_degenerate_basis (I I' : basis) :
  point_of_basis_pert I = point_of_basis_pert I' -> I = I'.
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Proof sketch

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Fact [...] : col (1+j) (point_of_basis_pert I) != 0 <-> (j \in I).
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Lemma no_degenerate_basis (I I' : basis) :
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Outcome

We arrive at a function `phase2 : lex_feasible_basis -> result`

- iterate over the lex-feasible bases;
- **terminates** in a finite number of steps;
- either finds an optimal lex-feasible basis,
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```
Inductive result :=
| Optimal_basis of lex_feasible_basis.
| Unbounded_cert (I: lex_feasible_basis) of 'I_#|I|
```

```
Lemma phase2P : forall I_0, phase2_spec (phase2 I_0).
```

where

```
Inductive phase2_spec : result -> Type :=
| Optimal (I: lex_feasible_basis) of
  (reduced_cost I) >=m 0 : phase2_spec (Optimal_basis I).
| Unbounded (I: lex_feasible_basis) (i: 'I_#|I|) of
  (reduced_cost I) i 0 < 0 /\ (A *m direction I i) >=m 0 :
    phase2_spec (Unbounded_cert i).
```

Phase I simplex method : finding an initial feasible basis

Usually handled by solving an auxiliary LP:

- with a trivial feasible basis;
- on which we call Phase II;
- which determines whether the LP is feasible or not.

Phase I Linear Program

$$\begin{aligned} \text{minimize} \quad & \langle e, y - A_K x \rangle \\ \text{subject to} \quad & A_K x \leq b_K + y\,,\ A_L x \geq b_L \\ & y \geq 0\,,\ (x, y) \in \mathbb{R}^{n+p} \end{aligned}$$

where K and L are built from an (arbitrary) basis I:

$$K := \{i \in [m] \colon A_i x^I < b_i\}\,, \qquad L := \{i \in [m] \colon A_i x^I \geq b_i\}\,.$$

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Problem

Lots of **painful** and **boring** manipulations of block matrices:

block structure reflected on bases and submatrices.

Alternative to Phase I

Our definition of emptiness predicate provides a witness point:

```
Theorem emptyPn A b : reflect (exists x, A *m x >=m b) (~~ empty A b).
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Idea

Find a (lex-)feasible basis from the feasible point `x`!

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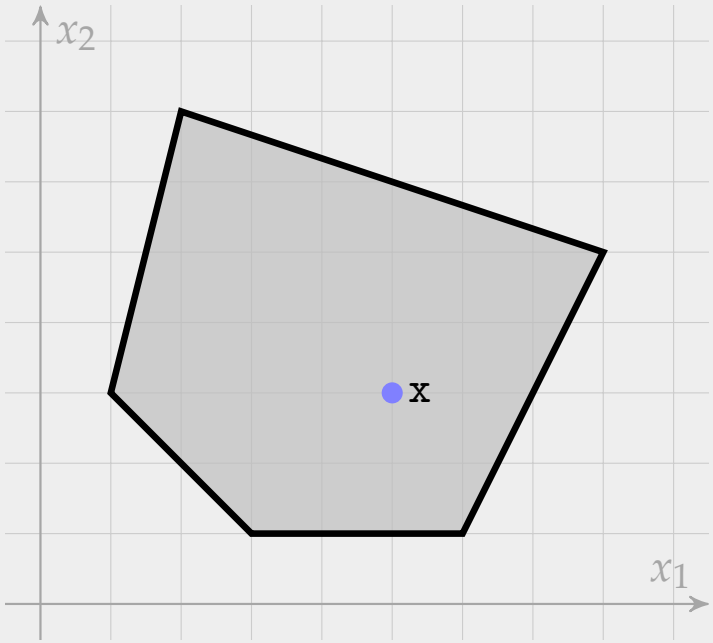
While $\mathsf{rank}\, A_{I(x)} < n$, do



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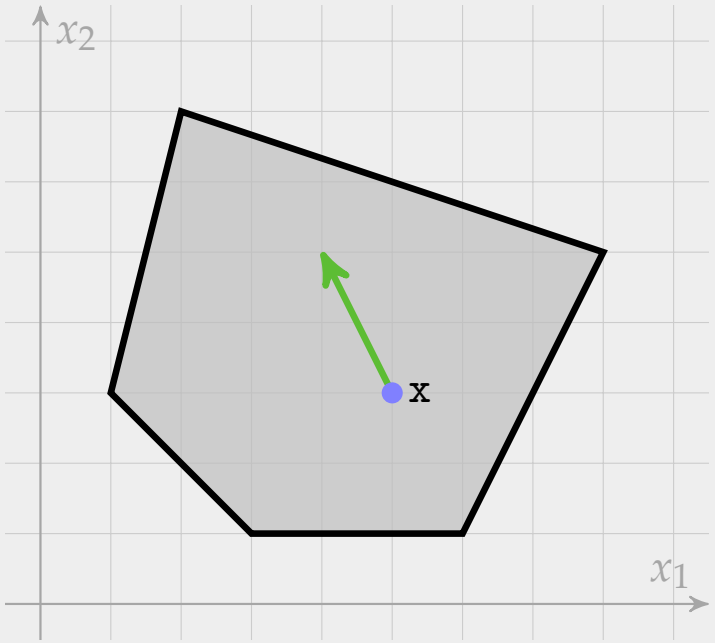

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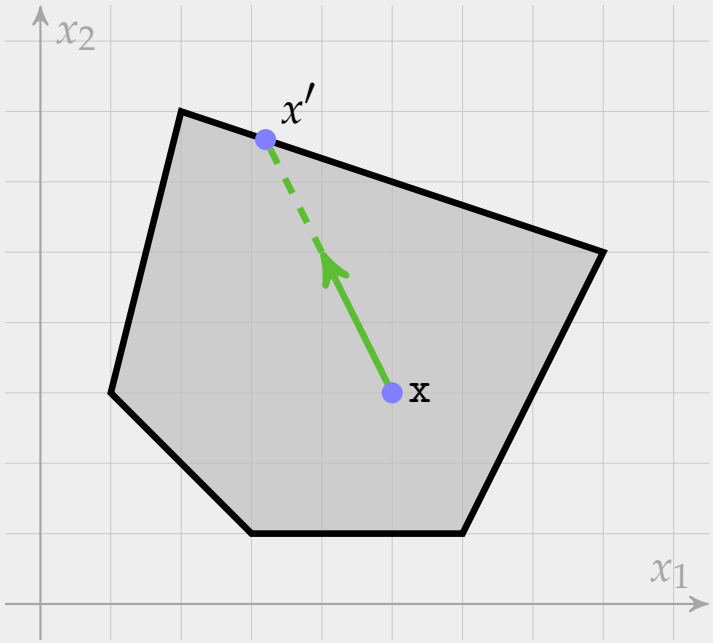

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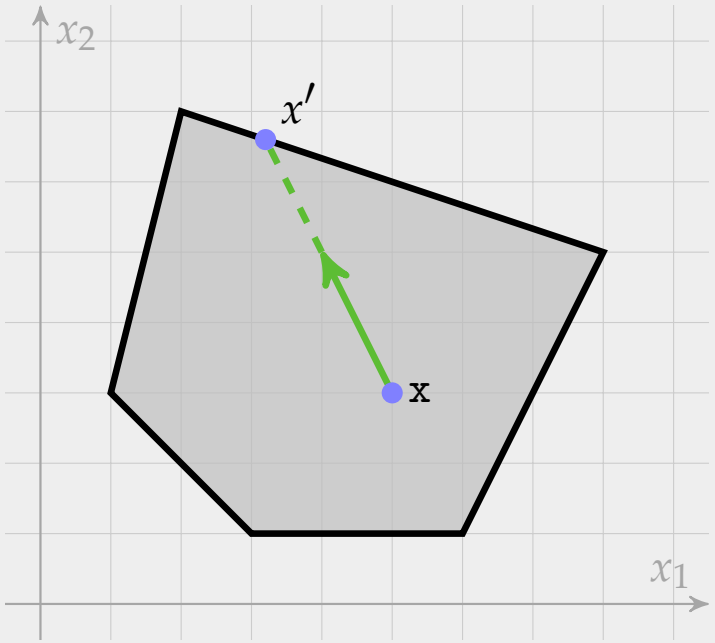

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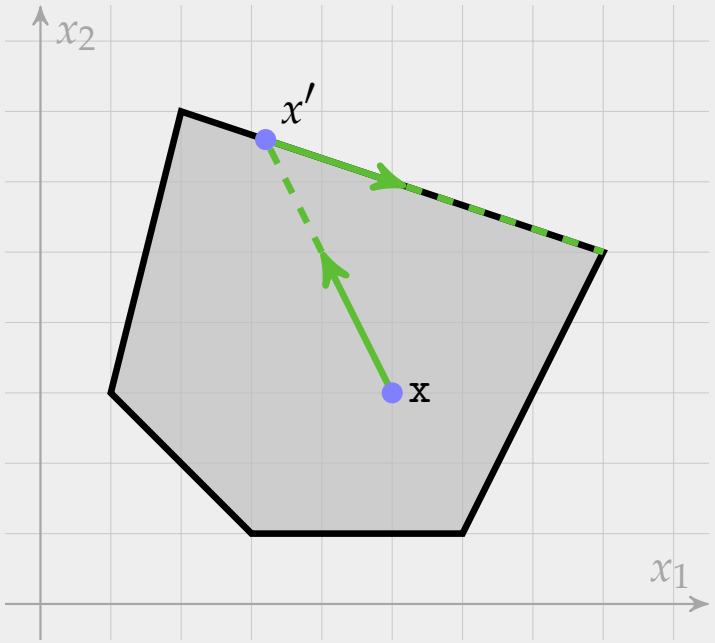

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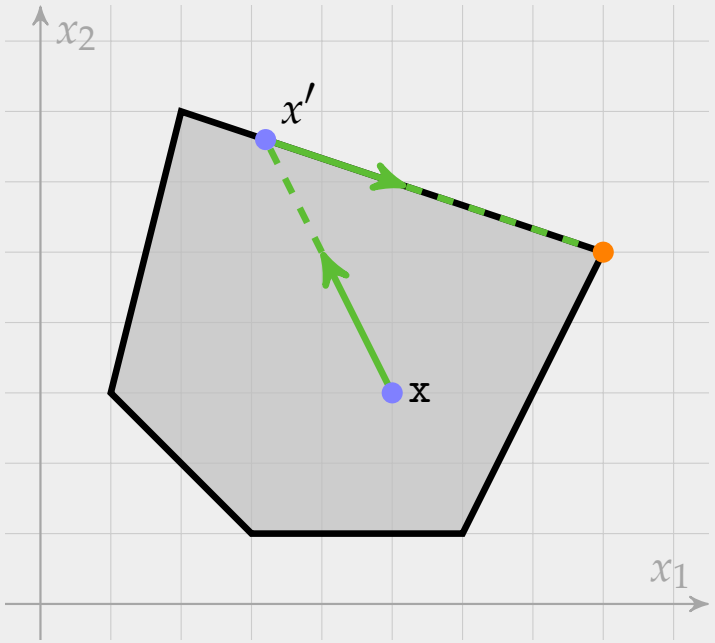

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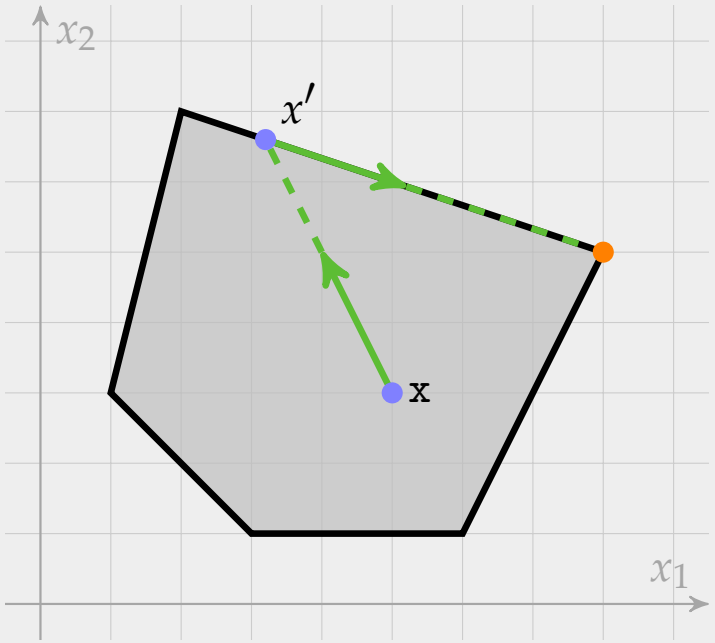


Benefits

- relies on the same ingredients as pivoting
- applies to the symbolic perturbation setting of lexicographic rule

Outline of the talk

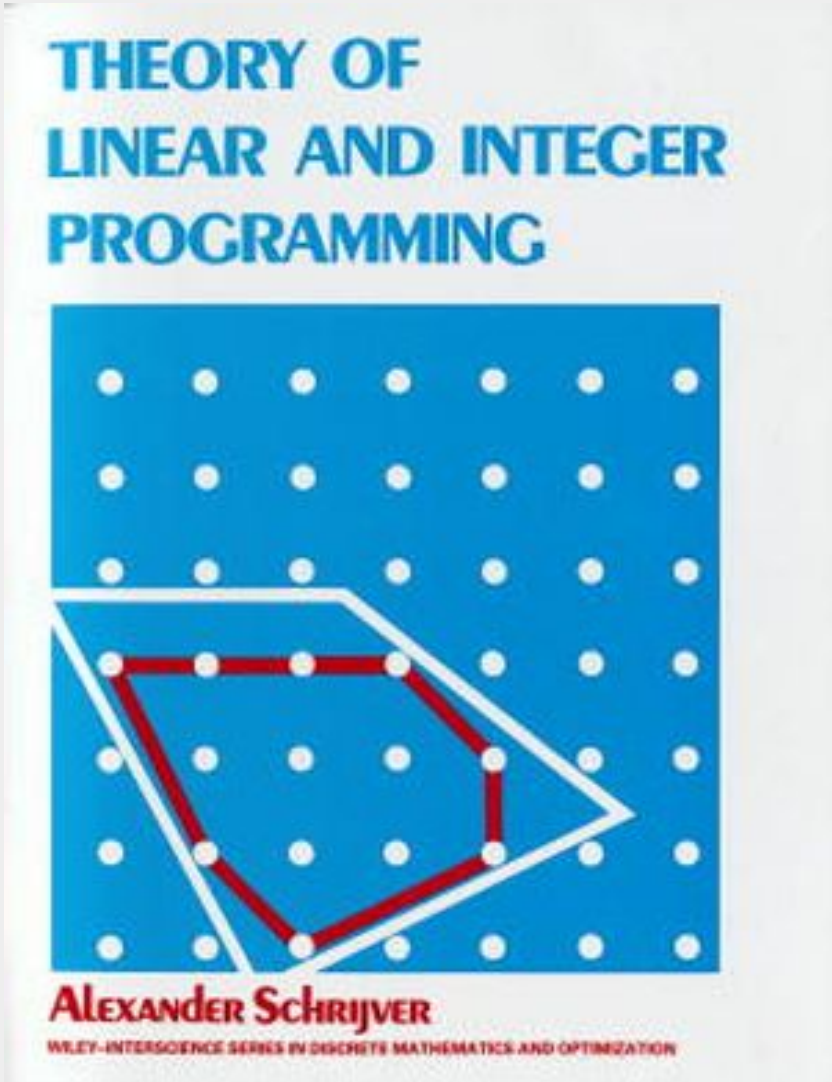
Concluding remarks

Contribution

Foundations of the formalization of convex polyhedra in Coq

- extensive use of Boolean reflection methodology
  $\Longrightarrow$ **constructive** formalization
  $+$ benefit from the Mathematical Components library
- formalization of the simplex method
- lexicographic rule yields a rather simple termination proof
  $+$ opens perspectives to enumerate the vertices (basic points)
- pragmatic way to deal with Phase I
- many essential results on polyhedra follows from the simplex method (with a constructive proof!)

Concluding remarks (2)



7

Fundamental concepts and results on polyhedra, linear inequalities, and linear programming

In this chapter we first state a fundamental theorem on linear inequalities (Section 7.1), and next we derive as consequences some other important results, like the Finite basis theorem for cones and polytopes, the Decomposition theorem for polyhedra (Section 7.2), Farkas' lemma (Section 7.3), the Duality theorem of linear programming (Section 7.4), an affine form of Farkas' lemma (Section 7.6), Carathéodory's theorem (Section 7.7), and results for strict inequalities (Section 7.8). In Section 7.5 we give a geometrical interpretation of LP-duality. In Section 7.9 we study the phenomenon of *complementary slackness*.

Each of the results in this chapter holds both in real spaces and in rational spaces. In the latter case, all numbers occurring (like matrix and vector entries, variables) are restricted to the rationals.

7.1. THE FUNDAMENTAL THEOREM OF LINEAR INEQUALITIES

The fundamental theorem is due to Farkas [1894, 1898a] and Minkowski [1896], with sharpenings by Carathéodory [1911] and Weyl [1935]. Its geometric content is easily understood in three dimensions.

Theorem 7.1 (Fundamental theorem of linear inequalities). *Let $a_1, \ldots, a_m$, b be vectors in n-dimensional space. Then:*

either 1. *b is a nonnegative linear combination of linearly independent vectors from $a_1, \ldots, a_m$;*

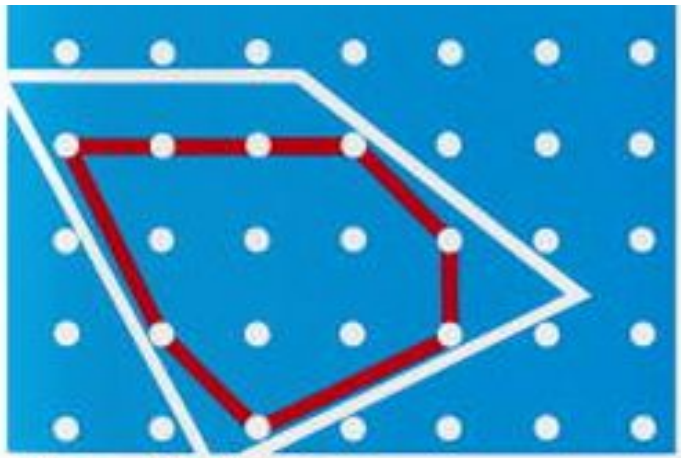
85

Concluding remarks (2)

THEORY OF

7

In this chapter we first state a fundamental theorem on linear inequalities (Section 7.1), and next we derive as consequences some other important results, like the Finite basis theorem for cones and polytopes, the Decomposition theorem for polyhedra (Section 7.2), Farkas' lemma (Section 7.3), the Duality theorem of linear programming (Section 7.4), an affine form of Farkas' lemma (Section 7.6), Carathéodory's theorem (Section 7.7), and results for strict inequalities (Section 7.8). In Section 7.5 we give a geometrical interpretation of LP-duality. In Section 7.9 we study the phenomenon of *complementary slackness*.



ALEXANDER SCHRIJVER

WILEY-INTERSCIENCE SERIES IN DISCRETE MATHEMATICS AND OPTIMIZATION

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The above proof of this fundamental theorem also gives a fundamental algorithm: it is a disguised form of the famous *simplex method*, with *Bland's rule* incorporated- -see Chapter 11 (see Debreu [1964] for a similar proof, but with a lexicographic rule).

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Concluding remarks (3)

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     spec    proof comments
      128      239        1 theories/extra_matrix.v
       57      105        0 theories/extra_misc.v
       57       77       10 theories/inner_product.v
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- write “good” proof: maintenance, factorization
- scaling up to large projects requires a lot of rewriting
- a lot of fun

Perspectives

Right now

Define a **type** for polyhedra: quotient external representations w.r.t. set equality

- equivalence is decidable
- but no canonical representation!

+ introduce the notion of faces, intrinsically related with the representation issue.

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Master internship / PhD: contact me if you're interested!

Thank you!

 `github.com/nhojem/Coq-Polyhedra`

arXiv:1706.10269

Georges Gonthier, Andrea Asperti, Jeremy Avigad, Yves Bertot, Cyril Cohen, François Garillot, Stéphane Le Roux, Assia Mahboubi, Russell O'Connor, Sidi Ould Biha, Ioana Pasca, Laurence Rideau, Alexey Solovyev, Enrico Tassi, and Laurent Théry. A machine-checked proof of the odd order theorem. In Sandrine Blazy, Christine Paulin-Mohring, and David Pichardie, editors, **Interactive Theorem Proving**, pages 163–179, Berlin, Heidelberg, 2013. Springer Berlin Heidelberg. ISBN 978-3-642-39634-2.

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