Solvers Principles and Architecture (SPA)

Convex Optimization

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- Find an **optimum** of a function over a **constrained set**
- Optimum: minimum or maximum
- The function to optimize is called the **objective** or **cost** function
- The constrained set is called the **feasible set**
- A solution to the problem is called an **objective vector**
- An optimum is called an **optimal value**

$$
\begin{array}{lll}
\min & f_0(x) & \\
s.t. & f_i(x) \leq 0, & i = 1, \ldots, m \\
 & h_j(x) = 0, & j = 1, \ldots, p
\end{array}
$$

- $x \in \mathbb{R}^n$: optimization variable (vector)
- $f_0 : \mathbb{R}^n \to \mathbb{R}$, objective or cost function
- $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$: inequality constraint functions
- $h_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, p$: equality constraint functions

Optimal value:

$$p^\star = \inf\{f_0(x) \mid f_i(x) \leq 0, i = 1, \ldots, m, h_j(x) = 0, j = 1, \ldots, p\}$$

- $p^\star = -\infty$: the problem is **unbounded** (below)
- $p^\star = \infty$: the problem is **infeasible** (no x satisfies the constraints)

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Optimization problem in standard form

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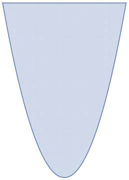
- The objective function is **convex**
- The feasible set is **convex**

What is convexity?

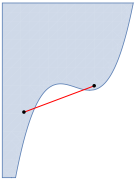
Outline

1. Introduction
2. Convexity
3. Convex optimization
4. Duality
5. Simplex algorithm
6. Interior point methods
7. Semidefinite Programming

- **Intuition**: A set C is convex if and only if, for any two points in C, the shortest path that links these two points is also entirely in C.
- In this course: we will restrict ourselves to the flat Euclidean space $\mathbb{R}^n$, thus $C \subseteq \mathbb{R}^n$.
- As a vector space, a point $p \in \mathbb{R}^n$ is a vector and one can define scalar multiplication and addition over vectors.
- In these settings, C is convex if and only if, for all $c_1, c_2 \in C$, for all $\lambda \in [0, 1], \lambda c_1 + (1 - \lambda) c_2$ is also in C.



Convex



Non convex

Definition: The **epigraph** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\text{epi}(f) := \{(x, y) \mid f(x) \leq y\} \subset \mathbb{R}^n \times \mathbb{R}$$

- f is **convex** if and only if its epigraph is a convex set
- f is **concave** if and only if $-f : x \mapsto -f(x)$ is convex

Examples:

- $f : x \mapsto x^2$ is convex (cf. left figure in the previous slide)
- $f : x \mapsto x^3 + x^2$ is not convex (cf. right figure in the previous slide)

- $\forall \lambda \in [0,1].\ \forall x, y. \quad f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$
- **Intuition**: the image of a point in the segment joining x and y is somewhere below the segment joining $f(x)$ and $f(y)$
- **Any local minimum of f is also a global minimum**
- One can define a weak notion of differentiability over convex functions
- The sub-differential of $f : \mathbb{R}^n \to \mathbb{R}$ at x is defined by the following set:

$$\partial f(x) := \{z \in \mathbb{R}^n \mid \forall t \in \mathbb{R}^n. \quad f(t) \geq f(x) + z \cdot (t - x)\}$$

  where $x \cdot y$ denotes the usual scalar product over $\mathbb{R}^n$
- **Intuition**: the sub-differential at x is the set of all affine functions that touches the graph of f only at x
- **Example**: the absolute value function is non-differentiable at 0 in the usual sense, but it is sub-differentiable, $\partial f(0) = [-1, 1]$

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Find the minimum of a convex function $f : \mathbb{R}^n \to \mathbb{R}$, over a convex set C

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & x \in C \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & Ax \leq b \qquad (\mathcal{P}) \\ & x \geq 0 \end{aligned}$$

- **This lecture**: C will be a polyhedron, or a linear convex set
- A is an $m \times n$ matrix of rank m (no redundant rows)
- Inequalities are component-wise
- x is in the positive orthant ($\mathbb{R}^n_+$)
- Any set of linear non-strict constraints can be put in this form: if $x \leq a$ then introduce $y \geq 0$ and substitute x by $a - y$ everywhere

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The Lagrangian associated to $(\mathcal{P})$

Intuition: inject the constraint into the objective function

The Lagrangian associated to $(\mathcal{P})$ is defined by ($y \in \mathbb{R}^m$, $y \geq 0$)

$$L(x, y) = f(x) + y \cdot (Ax - b)$$

If there exists an index i such that the ith component of the vector $Ax - b$ is positive, then $L(x, y)$ is unbounded since y_i can be chosen arbitrarily big. Thus

$$\sup_y L(x, y) = \begin{cases} f(x) & \text{if } Ax \leq b \\ +\infty & \text{otherwise} \end{cases}$$

Solving $(\mathcal{P})$ is then equivalent to minimizing $\sup_y L(x, y)$ over x:

$$\text{val}(\mathcal{P}) = \inf_x \sup_y L(x, y)$$

In general (no convexity assumptions are needed here), if L is a real valued function defined over the product $X \times Y$, then

$$\sup_y \inf_x L(x,y) \leq \inf_x \sup_y L(x,y)$$

Proof. Let $(\bar{x}, \bar{y}) \in X \times Y$, then, by definition of inf and sup

$$\inf_x L(x,\bar{y}) \leq L(\bar{x},\bar{y}) \leq \sup_y L(\bar{x},y)$$

So the quantity $\sup_y L(\bar{x},y)$ is an upper bound of $\inf_x L(x,\bar{y})$. Since the sup is the smallest upper bound by definition, one gets

$$\sup_y \inf_x L(x,\bar{y}) \leq \sup_y L(\bar{x},y)$$

But then $\sup_y \inf_x L(x,\bar{y})$ is a lower bound for $\sup_y L(\bar{x},y)$. Since, dually, the inf is the biggest lower bound, one gets the desired result.

Weak duality applied to $(\mathcal{P})$

By the weak duality, we get a lower bound of the objective value

$$\text{val}(\mathcal{D}) := \sup_y \inf_x L(x, y) \leq \inf_x \sup_y L(x, y) = \text{val}(\mathcal{P})$$

where $\text{val}(\mathcal{D})$ if the objective value of a distinct, yet related, optimization problem, $(\mathcal{D})$, called the dual problem, and defined by $\sup_y \inf_x L(x, y)$, for the exact same Lagrangian L of $(\mathcal{P})$, called the primal problem.

Suppose $f(x) = c \cdot x$ for some vector $c \in \mathbb{R}^n$.
The Lagrangian L could be rearranged as follows (recall that $Ax \cdot y = x \cdot A^t y$, where A^t denotes the transpose of the matrix A):

$$L(x, y) = -b \cdot y + x \cdot (A^t y + c)$$

Since $x \geq 0$, we get

$$\inf_x L(x, y) = \begin{cases} -b \cdot y & \text{if } A^t y + c \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

and hence, the dual problem is (the primal is recalled on the right)

$$\begin{aligned} \max \quad & -b \cdot y \\ s.t. \quad & A^t y \geq -c \qquad (\mathcal{D}) \\ & y \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \min \quad & c \cdot x \\ s.t. \quad & Ax \leq b \qquad (\mathcal{P}) \\ & x \geq 0 \end{aligned}$$

- The evaluation of the dual cost function on any feasible point of the dual problem bounds from below the evaluation of the primal cost function on any feasible point of the primal problem:

$$\forall y.\ A^t y \geq -c, \quad \forall x.\ Ax \leq b. \quad -b \cdot y \leq c \cdot x$$

- If the primal problem is degenerated then the dual is unfeasible
- If the dual problem is degenerated then the primal is unfeasible
- The primal and dual cannot be degenerated simultaneously, but
- The primal and the dual can be both unfeasible:

$$\begin{array}{lll} \max & y & \\ s.t. & 0y \geq 1 & (\mathcal{D}) \\ & y \geq 0 & \end{array} \qquad \begin{array}{lll} \min & -x & \\ s.t. & 0x \leq -1 & (\mathcal{P}) \\ & x \geq 0 & \end{array}$$

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Strong duality occurs when $\mathrm{val}(\mathcal{D}) = \mathrm{val}(\mathcal{P})$

Theorem

When the cost function is linear, $(\mathcal{P})$ and $(\mathcal{D})$ are either strongly dual or both unfeasible.

Proof. By Farkas' Lemma (upcoming lectures)

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Any polyhedron $Ax \leq b$ can be seen as the projection of a polyhedron $A'x' = b$ of higher dimension: simply augment the vector $x \in \mathbb{R}^n$ with the slack vector $s \in \mathbb{R}^m$, $s \geq 0$, and the matrix A with the identity matrix I_m of $\mathbb{R}^m$ to saturate all inequalities:

$$A' := \begin{pmatrix} A & I_m \end{pmatrix}; \quad x' := \begin{pmatrix} x \\ s \end{pmatrix} \in \mathbb{R}^{n+m}$$

Thus $A'x' = Ax + I_m s = b$.
(If x_i is unconstrained, one adds two nonnegative variables s_{1i}, s_{2i} and substitute x_i by $s_{1i} - s_{2i}$)

So there is no loss of generality if one considers the saturated form

$$A'x' = b \quad \text{instead of} \quad Ax \leq b$$

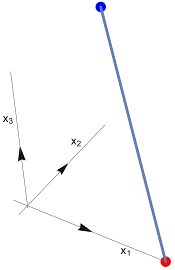
Consider the convex polyhedron $Ax = b$, where the rank of A is m

- If $m > n$, the problem is unfeasible
- Thus, in the sequel, we will suppose that $m \leq n$

Vertex of convex polyhedron. For $x \geq 0$, define

$$\mathfrak{B} := \{i \mid x_i > 0\} \qquad \mathfrak{N} := \{i \mid x_i = 0\}$$

Then, rearrange the matrix A accordingly: $A = \begin{pmatrix} A_{\mathfrak{B}} & A_{\mathfrak{N}} \end{pmatrix}$.
x is a vertex if $\text{rank}(A_{\mathfrak{B}}) \leq m$.



- degenerated vertex $\text{rank}(A_{\mathfrak{B}}) < m$
- non-degenerated vertex $\text{rank}(A_{\mathfrak{B}}) = m$
- **Example**: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- $(1, 0, 0)$ (red point) is degenerated
- $(0, 1, 1)$ (blue point) is non-degenerated

Let $\hat{x} = \begin{pmatrix} \hat{x}_{\mathfrak{B}} \\ \hat{x}_{\mathfrak{N}} \end{pmatrix}$ be a feasible non-degenerated vertex of $Ax = b$

- $\hat{x}_{\mathfrak{B}} > 0$, $\hat{x}_{\mathfrak{N}} = 0$ and $A_{\mathfrak{B}}$ is invertible (because $\hat{x}$ is non-degenerated)
- Let x be a feasible point in an open ball centered at $\hat{x}$ and separating $\hat{x}$ from all the other feasible vertices
- Such a ball is non-empty because vertices are discrete
- For all x in this ball the constraint $Ax = b$, rewritten as $A_{\mathfrak{B}} x_{\mathfrak{B}} + A_{\mathfrak{N}} x_{\mathfrak{N}} = b$, links $x_{\mathfrak{B}}$ to $x_{\mathfrak{N}}$, namely
- Thus $x_{\mathfrak{B}} = A_{\mathfrak{B}}^{-1}(b - A_{\mathfrak{N}} x_{\mathfrak{N}})$
- So locally, the cost function $c \cdot x$ depends only on $x_{\mathfrak{N}}$, indeed
- $c \cdot x = \begin{pmatrix} c_{\mathfrak{B}} \\ c_{\mathfrak{N}} \end{pmatrix} \cdot \begin{pmatrix} A_{\mathfrak{B}}^{-1}(b - A_{\mathfrak{N}} x_{\mathfrak{N}}) \\ x_{\mathfrak{N}} \end{pmatrix} = \underbrace{\left(c_{\mathfrak{N}} - A_{\mathfrak{N}}^{t} {A_{\mathfrak{B}}^{-t}} c_{\mathfrak{B}}\right)}_{r} \cdot x_{\mathfrak{N}} + \underbrace{c_{\mathfrak{B}} \cdot A_{\mathfrak{B}}^{-1} b}_{a}$
- Thus, $c \cdot x$ has the form $r \cdot x_{\mathfrak{N}} + a$
- To locally decrease the value of $c \cdot x$, only $x_{\mathfrak{N}}$ counts

- The function $r \cdot x_{\mathfrak{N}}$ is called the reduced cost function
- We seek a displacement that locally decreases the reduced cost function
- Suppose that there exists a index j such that the jth component of the vector r is negative, $r_j < 0$
- Consider a displacement along e_j (where e_j denotes jth vector of the canonical orthonormal basis of $\mathbb{R}^{n-m}$), ρ is a positive real number
- $r \cdot (\hat{x}_{\mathfrak{N}} + \rho e_j) = r \cdot \hat{x}_{\mathfrak{N}} + \rho r \cdot e_j = r \cdot \hat{x}_{\mathfrak{N}} + \rho r_j < r \cdot \hat{x}_{\mathfrak{N}}$, and
- $\hat{x}_{\mathfrak{N}} + \rho e_j \geq 0$ (feasible)

Optimality criterion: $r \geq 0$
No possible local improvement for $r \cdot x_{\mathfrak{N}}$, hence we reached a local minimum, which is also global by convexity.

- The displacement ρ could be bounded
- Recall that locally $x_{\mathfrak{B}} = A_{\mathfrak{B}}^{-1}(b - A_{\mathfrak{N}} x_{\mathfrak{N}})$
- So the update $x_{\mathfrak{N}} \leftarrow \hat{x}_{\mathfrak{N}} + \rho e_j$ leads to $x_{\mathfrak{B}} \leftarrow \hat{x}_{\mathfrak{B}} + \rho d_{\mathfrak{B}}$, where $d_{\mathfrak{B}} = -A_{\mathfrak{B}}^{-1} A_{\mathfrak{N}} e_j$
- Since $x_{\mathfrak{B}} \geq 0$, we get $\hat{x}_{\mathfrak{B}} \geq \rho(-d_{\mathfrak{B}})$
- So ρ should not exceed the minimum of the quotient of the ith component of $\hat{x}_{\mathfrak{B}}$ over the $i$the component of $-d_{\mathfrak{B}}$ (whenever defined)

Unboundedness criterion: $d_{\mathfrak{B}} \geq 0$
In this case, ρ can be chosen arbitrarily big and the minimum is $-\infty$

- As seen earlier, a basis corresponds to a vertex (degenerated or not)
- When $\hat{x}$ is updated, one reaches a new basis $\mathfrak{B}$: one component of $x_{\mathfrak{B}}$ will vanish and gets replaced by a component of $x_{\mathfrak{N}}$ which becomes strictly positive after the update
- So an update moves from one vertex to another
- Moreover, such updates make $\hat{x}$ moves along the edges of the convex polyhedron $Ax = b$ since the displacement of $x_{\mathfrak{N}}$ are performed along one component at a time (namely the jth for the update $x_{\mathfrak{N}} \leftarrow \hat{x}_{\mathfrak{N}} + \rho e_j$)
- Recall: an edge is a facet of dimension 1

1. Start at a vertex (basis)
2. If the optimality criterion is satisfied, halt: the problem is solved
3. Otherwise, move along an edge that minimizes the reduced cost function
4. If the unboundedness criterion is satisfied, halt: the problem is unbounded
5. Otherwise, we reach a new basis and we loop back to the first step

Does it always terminate?

$$\begin{aligned} \min \quad & x_1 + x_2 \\ s.t. \quad & x_1 + x_2 = 0 \\ & x \geq 0 \end{aligned}$$

- Start with the basis $\mathfrak{B} = \{1\}$, $\mathfrak{N} = \{2\}$
- $\hat{x}^t = \begin{pmatrix} 0 & 0 \end{pmatrix}$, $A_{\mathfrak{B}} = A_{\mathfrak{N}} = \begin{pmatrix} 1 \end{pmatrix}$
- $r = c_{\mathfrak{N}} - A_{\mathfrak{N}}^t A_{\mathfrak{B}}^{-t} c_{\mathfrak{B}} = \begin{pmatrix} -1 \end{pmatrix}$ and $d_{\mathfrak{B}} = -A_{\mathfrak{B}}^{-1} A_{\mathfrak{N}} e_j = \begin{pmatrix} -1 \end{pmatrix}$
- update $x_{\mathfrak{N}} \leftarrow 0 + \rho$, $x_{\mathfrak{B}} \leftarrow 0 - \rho$, so $\rho = 0$
- So the algorithm is updating the basis without changing the vertex
- Similarly, the algorithm can cycle forever between several bases without actually halting

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