Master Sciences Informatiques

Solvers Principles and Architectures (SPA)

Final Exam, Fall 2018

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1 SAT/SMT Solvers

A. Quadratic Diophantine Equations

Consider the following quadratic Diophantine equation

$$ax^2 + by = c$$

where (x, y) are unknown integers and a, b, and c are three positive integers.

1. Prove that 3-SAT is NP-complete. Recall that 3-SAT is a restricted version of SAT where each clause contains exactly three literals.
2. Using 3-SAT, prove the existence of solutions of the above quadratic equation is also NP-completeness.

B. UNSAT Certificates

Suppose that your preferred SAT solver answers UNSAT for a given problem.

1. What options do you have to actually verify the veracity of such an output ?
2. Explain how Clause learning can be used to actually extract an UNSAT certificate, that is a (logical) proof of non satisfiability that can be checked by a human or a proof assistant.
3. How to extend such procedure to SMT?

2 Convex Optimization

A. Convexity

Is the following set is convex ?

$$\left\{\alpha \in \mathbb{R}^k \mid p(0) = 1, |p(t)| \leq 1 \text{ for } a \leq t \leq b\right\},$$

where $p(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_{k-1} t^{k-1}$.

B. Linear Programming

Let $A_i \in \mathbb{R}^m$, $i = 1, \ldots, n$. A cone generated by the set of vectors $\{A_1, \ldots, A_n\}$ is the following set

$$C := \left\{\sum_{i=1}^{n} x_i A_i \mid x_i \geq 0, i = 1, \ldots, n\right\} \subset \mathbb{R}^m \quad .$$

Notice that one can form a matrix A having the vectors A_i as its columns. In which case, the set C simply becomes $\{Ax \mid x \geq 0\}$.

A vector $b \in \mathbb{R}^m$ can be either inside the cone C or outside of it. Farkas' lemma says that the latter case is equivalent to the existence of a hyperplane with a normal vector $\mu \in \mathbb{R}^m$ that separates the vector b from the cone, formally:

$$\exists \mu \in \mathbb{R}^m. \quad A^t \mu \geq 0 \wedge b \cdot \mu < 0 \quad .$$

1. Take some time (less than 10mn) to try to prove the lemma (Bonus question)

2. Using Farkas' lemma, prove that strong duality holds in LP (except when both problems are unfeasible).

B. Duality

Consider the following optimization problem:

$$\begin{aligned} \min \quad & \frac{1+\cos(x)}{-2+\cos(x)} \\ \text{s.t.} \quad & \cos(x) \le 0 \\ & x \in \mathbb{R} \end{aligned}$$

1. Is this problem convex ?
2. What are the extremal points of the objective function when the constraint is discarded ? Which ones are feasible ? State simply why, in general for a constrained problem, an optimum may not be extremal in the usual sense.
3. Solve the optimization problem (cf. to Figure 1)
4. State the Lagrangian as well as the dual problem.
5. Are all KKT conditions satisfied ? Comment.
6. Compute the duality gap. (The exact numerical value is expected)

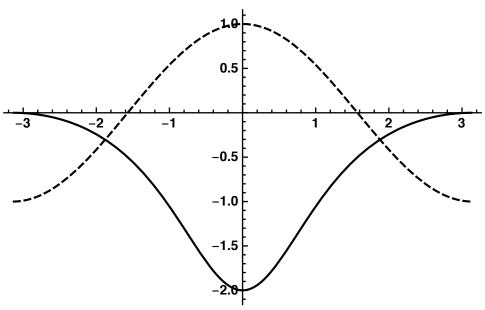

Figure 1: Plot of the objective function between $[-\pi, \pi]$. The cosine function is also given for convenience