Master Sciences Informatiques

Solvers Principles and Architectures (SPA)

Final Exam, Fall 2017

Khalil Ghorbal

1 SAT/SMT Solvers

A. CNF vs DNF

We have seen that converting any boolean (well formed) formula to an equivalent conjunctive normal form (CNF) increases linearly the number of logical connectives using Tseytin transformations.

1. Is this possible for disjunctive normal forms (DNF)?
2. Explain the reason for this asymmetry?

B. Resolution Rule

The resolution rule allows to eliminate via equivalent satisfiability a variable that appears both positively and negatively in different clauses. Assuming infinite memory, if one is to apply the original Davis/Putnam (DP) method to a given CNF formula till saturation (that is till reaching a fixed point).

1. What are the possible results of the algorithm and why?
2. If the formula is UNSAT, how can we extract a certificate of unsatisfiability?
3. How does such certificate relate to Craig interpolants?

C. CDCL

Conflict-Driven Clause Learning (CDCL) smartly reuses the resolution rule to prune the search tree built as a result of the original splitting rule suggested by DLL in 1962. Consider the following CNF formula

$$\begin{aligned}\phi &= c_1 \wedge c_2 \wedge c_3 \wedge c_4 \wedge c_5 \wedge c_6 \\ &= (x_5 \vee x_6) \wedge (x_1 \vee x_8 \vee \neg x_2) \wedge (x_1 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_4 \vee \neg x_5) \wedge (x_9 \vee \neg x_4 \vee \neg x_6)\end{aligned}$$

Assume the following decision assignments have been already made $x_9 = 0@2$ and $x_8 = 0@3$ and that the current decision assignment is $x_1 = 0@5$ (where the notation $x = b@n$ means that the variable x is assigned the value b at depth n).

1. Build the resulting implication graph.
2. Suggest a clause to learn from the observed conflict.
3. Prove that augmenting ϕ with such a formula is SAT equivalent to the original problem ϕ.
4. Based on the learned clause, at what depth should you backtrack?

D. SAT Reduction

The multiprocessing scheduling problem asks the following question. Given a finite set A of tasks, a measure (or time length) $\ell(a)$ for each task $a \in A$, a number m of processors and a deadline D, is there a partition $A = A_1 \cup A_2 \cup \cdots \cup A_m$ of A into m disjoint sets such that

$$\max_{1 \leq i \leq m} \left\{ \sum_{a \in A_i} \ell(a) \right\} \leq D \quad ?$$

Prove that the multiprocessing scheduling problem is NP-complete.

2 Convex Optimization

A. Linear Programming

The diet problem can be stated as follows: choose quantities $x_1, \ldots, x_n$ of n foods to find the cheapest healthy diet such that ($\imath$) one unit of food j costs c_j and contains amount a_{ij} of nutrient i, and ($\imath\imath$) a healthy diet requires nutrient i in quantity at least b_i.

1. Formulate the problem as a Linear Program (LP).
2. What is the interpretation (meaning) of the dual variables in this case? (write down the dual problem and comment.).

B. Simplex vs interior point methods

We want to solve the following problem, where f is twice continuously differentiable and assuming the primal objective value is finite and attained:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & Ax = b, \quad (\text{rank}\, A = p) \end{array}$$

1. Using the KKT conditions, deduce the optimality conditions on $x^\star$ and $\nu^\star$ (the Lagrange multiplier vector).
2. Suppose $\hat{x}$ is a solution for $Ax = b$, eliminate the equality constraint from the above problem.
3. Let $z^\star$ denote the optimal vector of this new unconstrained problem, deduce how $x^\star$ and $\nu^\star$ from $z^\star$.
4. Suppose f is linear, explain briefly the descent method used by the simplex algorithm with respect to the unconstrained problem and contrast it with the descent methods used in the interior point methods.

C. Duality for non convex problems

The two-way partitioning problem is stated as follows ($x \cdot y$ denotes the usual scalar product, W is a square matrix):

$$\begin{array}{ll} \min & x \cdot Wx \\ \text{s.t.} & x_i^2 = 1, \quad i = 1, \ldots, n \end{array}$$

1. Is this a convex problem (explain)?
2. Compute its Lagrangian and state its dual problem while classifying it (QP, LP, SDP, etc.).
3. Deduce a lower bound for the primal optimal value $p^\star$.

D. "Visualizing" symmetric positive semidefinite matrices

Recall that a symmetric matrix S is positive semidefinite if for all vectors z, the scalar product of z and Sz is nonnegative.

1. Define the set on which vary the components of the matrix S as a quantifier elimination problem.
2. Solve the problem for $n = 1$, n being the dimension of z. (the case $n = 2$ is depicted below)