Exact Certificates for Noncommutative Polynomial Optimization

Research Context. Certification and validation of computational results is a major issue for modern sciences raising challenging problems at the interplay of mathematics and computational aspects of computer science. One can emphasize in this context several applications arising in the design of modern cyber-physical systems with a crucial need of *exact* certification. In particular, one tries to avoid incidents such as the Patriot missile crash in 1991, the FDIV Pentium bug in 1994 or more recently the collision of Google's self-driving car in 2016. Since these systems often involve nonlinear functions, such as polynomials, it is highly desirable to design exact polynomial optimization methods. In 2001, Lasserre introduced a hierarchy of convex relaxations for approximating polynomial infima. Each relaxation is solved with a semidefinite programming (SDP) solver, implemented in finite-precision, thus providing only *approximate* certificates. The emergence of this exciting new field goes back to the last decade and has led to striking developments from a cross fertilization between (real) algebraic geometry, applied mathematics, theoretical computer science and engineering.

This project intends to design hybrid symbolic/numeric algorithms to output *exact* certificates for optimization problems with polynomial data, involving noncommutative (NC) variables, e.g. matrices. NC polynomials allow to handle certain control systems (e.g. orbit problems or H-∞ control) and quantum computation problems, (e.g. violation level of bell inequalities [8] or bosonic energy computations [4]). That is, one relies on expressions obtained by adding and multiplying matrices. A wide class of engineering problems has motivated major developments in this recently emerging area [2]. Connections between linear control systems may produce highly complicated systems involving polynomial matrix coefficients. These hard systems can be analysed thanks to an NC variant of the Lasserre's hierarchy [1]. This hierarchy shares the same interesting properties as in the commutative case, but comes also with certification and scalability issues.

The challenging goal of this internship is to design algorithms to compute *exact* certificates for noncommutative polynomial optimization problems, while controlling the bit complexity of the algorithmic procedures.

Goals Preliminary work will consist of studying the exisiting algorithms to obtain exact decompositions of nonnegative polynomials. In the commutative case, univariate polynomials have been recently studied in [7] by means of classical techniques from symbolic computation (root isolation, square-free decomposition). An extension to multivariate polynomials has been derived in [5] thanks to a perturbation/compensation algorithm.

A promising strategy is to adapt suitably this algorithm to the NC case. After designing the certification framework, further efforts should lead to provide the related bit complexity estimates, both on runtime and output size. Practical experiments shall be performed through implementing a tool within the Maple library `RealCertify` [6]. One expected goal is to compare the performance of the tool with existing frameworks, e.g. the rationalization scheme (rounding and projection algorithm) developed in [3].

Working Context The internship will be advised by Victor Magron (CNRS L2S CentraleSupelec / CNRS LAAS). The Master student will be hosted by the Mac team in the LAAS laboratory, located at Toulouse. The research shall be pursued in collaboration with Markus Schweighofer (University of Konstanz) and Igor Klep (University of Auckland), who are internationally renowned experts in noncommutative polynomial optimization .

Required Skills Motivated candidates should hold a Bachelor degree and have a solid background in **either** optimization, real algebraic geometry or computer algebra. Good programming skills are also required. The candidates are kindly asked to send an e-mail with "M2 candidate" in the title, a CV and motivation letter to `vmagron@laas.fr`. Knowledge of French does not constitute a pre-requisite.

A related PhD topic can be foreseen.

References

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