Modeling Physics with Differential-Algebraic Equations

Lecture 1

General Introduction to Differential Equations

Master Cyber-Physical Systems (M2)

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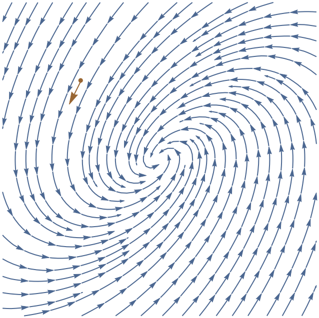
Functional equation with derivatives

$$(x', y') = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (\dot{x}, \dot{y}) = (-y, x - y)$$

Local description of motion



Ordinary (or Total) vs Partial ($\frac{\partial}{\partial u}$)

Sophus Lie

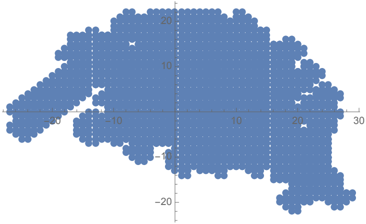
"Among all of the mathematical disciplines the theory of differential equations is the most important(...) It furnishes the explanation of all those elementary manifestations of nature which involve time."

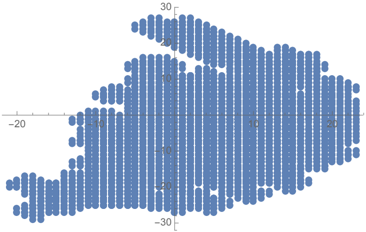
Convenient modeling language

- Continuous dynamics (vs discrete)
- No Boundary conditions (entire space)
- No memory (next state completely determined from the current)

An ant starts somewhere on a black and white squared plane

- if the square is white, the ant turns right then move forward
- if the square is black, the ant turns left then move forward
- the ant flips the color of its square before moving





- Cauchy-Lipschitz theorem: Local existence and unicity theorem (assuming Lipschitz continuity) $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in $X \subset \mathbb{R}$ if and only if there exists $K \geq 0$ such that

$$|f(y) - f(x)| \leq K|y - x|, \quad \forall x, y \in X$$

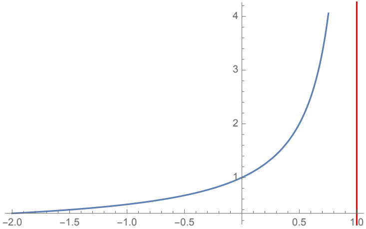
- Solutions often involve transcendental functions (sine, exp, etc.) For instance the first-order homogeneous equation $y' = ay$:

$$y = y_0 \exp(at)$$

- Liouville theorem: No closed form solutions in general $x' = \exp(-t^2)$ then

$$x(t) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt$$

- $x' = x^2$, $x(0) = x_0$ (Only locally Lipschitz)
- $x(t) = \frac{1}{\frac{1}{x_0} - t}$
- Singularity at $t = \frac{1}{x_0}$, maximum interval $(-\infty, \frac{1}{x_0})$



Numerical Integration

Euler Integration Schemes $x' = f(x)$

$$
\begin{array}{ll}
x^\bullet = x + f(x)\delta & \text{Explicit} \\
x^\bullet = x + f(x^\bullet)\delta & \text{Implicit}
\end{array}
$$

Other similar Integration Schemes: the Runge-Kutta family

Picard Iterations

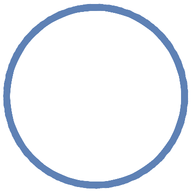
$$x^\bullet = x + \int_0^\delta f(x)dt$$

It boils down to approximate the integral term

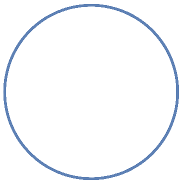
Numerical Analysis

- **Convergence**: does the numerical scheme approximates the solution when the discrete step goes toward zero ? The **order** gives the local quality of convergence.
- **Stability**: the propagation of errors (stiffness).

$$(x', y') = (-y, x)$$



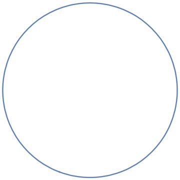
Euler (order 1)



Runge-Kutta (order 4)

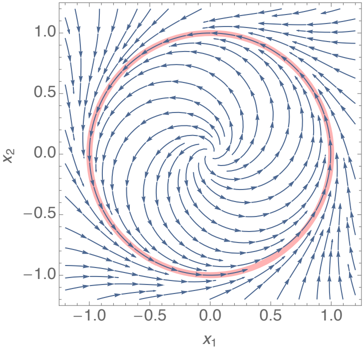
- **Geometrical Integration**: invariant-aware integration (e.g. Symplectic Methods)
- **Quantized State Systems** (QSS) Methods: efficient when simulating sparse systems

$$(x', y') = (-y, x)$$



Symplectic Integration

$$(\dot{x}_1, \dot{x}_2) = (x_1 - x_1^3 - x_2 - x_1 x_2^2, x_1 + x_2 - x_1^2 x_2 - x_2^3)$$



Algebraic Invariant Equation

The solution for $\mathbf{x}_0 = (1, 0)$ respects $\boxed{x_1(t)^2 + x_2(t)^2 - 1 = 0}$ $\forall t$

Numerical Integration & Qualitative Analysis

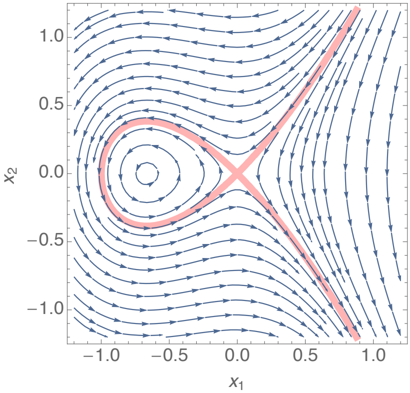
- More precise numerical integration (Geometrical Integration)
- Better understanding of the dynamics without solving the problem (some invariants represent conserved quantities like momentum or energy)

Formal Verification

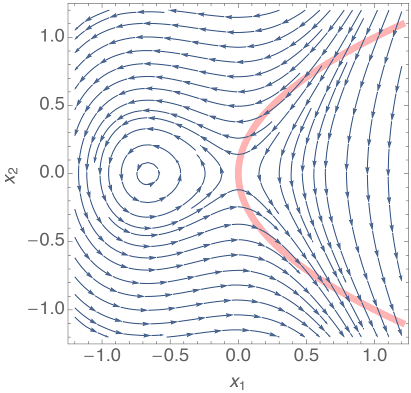
- Formal verification for dynamical and hybrid systems
- Static Analysis (as templates to statically analyze an implementation)
- Safety, Reachability, Stability

Problem I. *Checking Invariance of Algebraic Equations*

Given $\dot{\mathbf{x}} = (-2x_2, -2x_1 - 3x_1^2)$, $p(\mathbf{x}_0) = 0$, is $p(\mathbf{x}(t)) = 0$ for all t ?



$$p(x_1, x_2) = x_1^2 + x_1^3 - x_2^2$$



$$p(x_1, x_2) = x_1 - x_2^2$$

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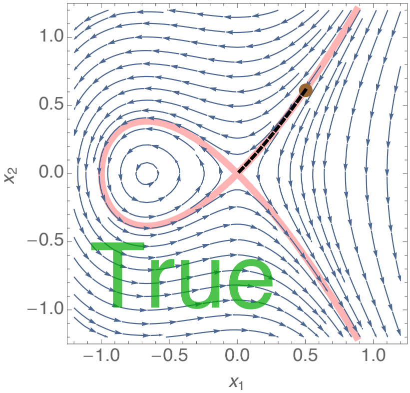

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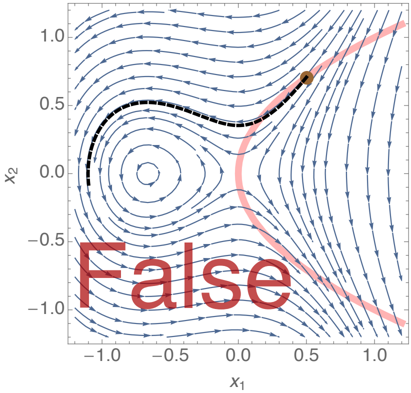
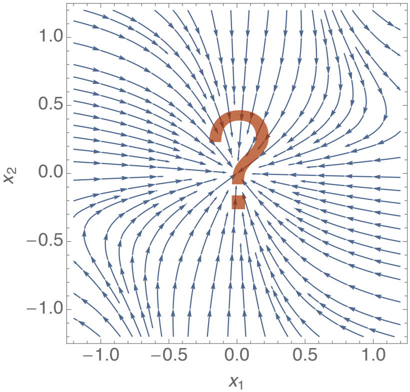


$p(x_1, x_2) = x_1 - x_2^2$

Problem II. *Generate Algebraic Invariant Equations*

Given $\dot{\mathbf{x}} = (-x_1 + 2x_1^2 x_2, -x_2)$, how to generate p such that $p(\mathbf{x}(t)) = 0$?



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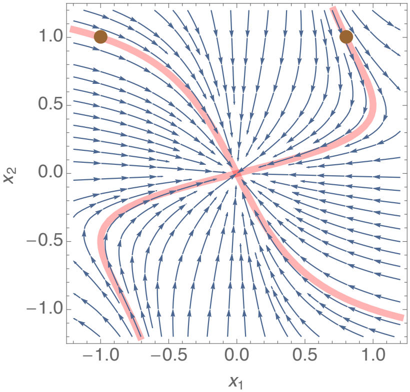

$$p_{(x_1(0),x_2(0))}(x_1, x_2) = (x_2(0) - x_1(0)x_2(0)^2)x_1 - x_1(0)(x_2 - x_1 x_2^2) = 0$$

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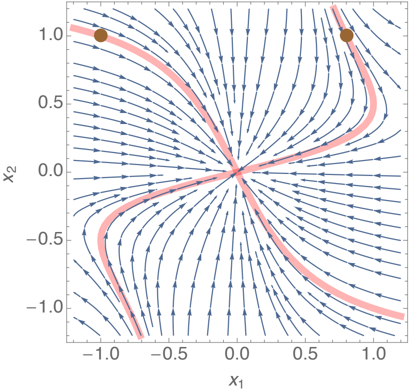

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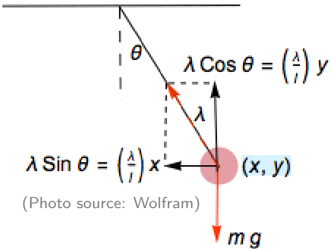
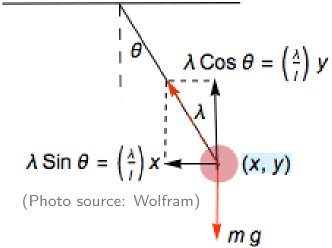
$\frac{x_1}{x_2 - x_1 x_2^2}$ is an invariant **rational function**.

1. Ordinary Differential Equations:
   - Cauchy-Lipschitz theorem: existence and uniqueness of solutions
   - Liouville theorem: no closed form solutions in general
   - Numerical integration: convergence and stability
   - Qualitative analysis: invariant regions

2. Next: Differential-Algebraic Equations (Examples)

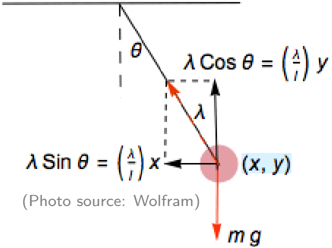


(Photo source: Wolfram)



(Photo source: Wolfram)

$$
\begin{aligned}
\ddot{x} &= -\lambda x \\
\ddot{y} &= -\lambda y - g && \text{(Newton's law)} \\
0 &= L^2 - x^2 - y^2 && \text{(Algebraic constraint)}
\end{aligned}
$$



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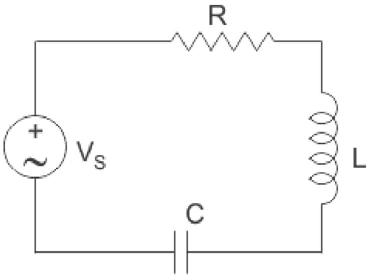
State variables: $(x, y, \dot{x}, \dot{y})$: $\begin{cases} x, y & \textbf{differential} \text{ variables} \\ \lambda & \textbf{algebraic} \text{ variable} \end{cases}$

Lagrange Equations

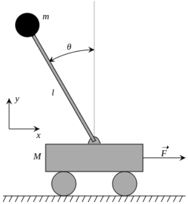
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q + F^t \lambda$$

- Lagrangian: $L = T - U$ (Kinetic and potential Energies)
- Generalized coordinates $q = (q_1, \ldots, q_n)$
- Holonomic constraints: $f(q) = 0$
- Nonconservative forces: Q
- F^t: the transpose of the Jacobian of f
- λ: vector of Lagrange multipliers



$$\dot{V}_C = \frac{I}{C}$$
$$\dot{I} = \frac{V_L}{L}$$
$$0 = V_R - RI \quad \text{Ohm's Law}$$
$$0 = V_S - V_R - V_L - V_C \quad \text{Algebraic Constraint}$$

State variables: (V_R, V_C, V_L, I)



(Photo source: Wikipedia)

State variables: (θ, x)

$$L = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 - mg\ell\cos(\theta)$$

$$v1 = \left(\frac{d}{dt}x, 0\right)$$

$$v2 = \left(\frac{d}{dt}(x - \ell\sin(\theta)), \frac{d}{dt}(\ell\cos(\theta))\right)$$

Lagrange Equations

$$\begin{aligned} F &= (M+m)\ddot{x} - m\ell\ddot{\theta}\cos(\theta) + m\ell\dot{\theta}^2\sin(\theta) \\ \ddot{x}\cos(\theta) &= \ell\ddot{\theta} - g\sin(\theta) \end{aligned}$$

Control Problem: Find F such that

$$\theta \in [\theta_r - \epsilon, \theta_r + \epsilon]$$

for some given reference value θ_r

- Non-Linear (inverted pendulum):

$$f(\dot{x}, x, t) = 0, \qquad (f \text{ nonlinear})$$

- Linear (RLC circuit):

$$A(t)\dot{x} + B(t)x + c(t) = 0$$

.

- Semi-Explicit (pendulum):

$$\begin{cases} \dot{x} & = f(x, y, t) \\ 0 & = g(x, y, t) \end{cases}$$

.

Next Lecture: More on DAEs

- Index reduction
- Numerical integration
- Modelling tools

Some References

- Wolfgang Walter: *Ordinary Differential Equations*. Springer New York, 1998
- Ernst Hairer, Christian Lubich, Gerhard Wanner: *Geometric Numerical Integration*. Springer Berlin Heidelberg, 2009
- Peter Kunkel, Volker Mehrmann: *Differential-Algebraic Equations*. European Mathematical Society, 2006