Exam 2022-2023 (Solutions)

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We established a formal link between typed λ-calculi and cartesian closed categories (Curry-Howard-Lambek isomorphism). But what if we remove types? To which structure the pure (untyped) λ-calculus is isomorphic? Let's see!

Recall that a (small) category with one object is a monoid, that is a semigroup with a unitary element. (recall that small means that the class of morphisms between any two elements is a set. It is a technicality that you can drop in what follows.)

0. Describe a small closed cartesian category with only one object (elements and arrows are to be made explicit).

0. sol a cartesian closed category has a terminal object, thus the unique object of the monoid is that terminal object, which necessarily has a unique identity reflexive map. Thus we have the simplest possible non-empty category that has one element and one arrow and which satisfies all the axioms of a ccc.

A C-monoid (for Curry or Cartesian) is a monoid $\mathscr{M}$ that is almost a cartesian closed category in the sense that it enjoys similar structure. Technically $\mathscr{M}$ has an extra structure $(\pi, \pi', \epsilon, *, <>)$, where π, π', and ϵ are elements (or nullary operations) of $\mathscr{M}$, $(-)^*$ is a unary operation and $< -, - >$ is a binary operation satisfying the following identities for all a, b, c, h, and k (where composition is denoted by simple concatenation, so ab denotes $a \circ b$):

(A1.) $\pi < a, b >= a$

(A2.) $\pi' < a, b >= b$

(A3.) $< \pi c, \pi' c >= c$

(A4.) $\epsilon < h^* \pi, \pi' >= h$

(A5.) $(\epsilon < k\pi, \pi' >)^* = k$

1. Recall briefly what do these axioms refer to in a standard cartesian closed category? (observe that the type subscript is omitted since there exists only one element.)

1. sol If we had a cartesian closed category, π and π' would be the projections, ϵ the evaluation mapping (for the exponential), and the star $(*)$ is the transposition.

1.' What's missing to have a full cartesian closed category?

1.' sol The terminal object.

2. Using the above axioms, prove the following consequences for all a, b, c, h, and k

(A6.) $< a, b > c =< ac, bc >$

(A6. sol) $< a, b > c =< \pi < a, b > c, \pi' < a, b > c >=< ac, bc >$

(A7.) $<\pi,\pi'>=1$ by A3, A1 and A2.

(A7. sol) $<\pi,\pi'>=<\pi 1,\pi' 1>=1$ by the identity law and A3.

(A8.) $\epsilon<h^*a,b>=h<a,b>$

(A8. sol) $h<a,b>=\epsilon<h^*\pi,\pi'><a,b>=\epsilon<h^*\pi<a,b>,\pi'<a,b>>=\epsilon<h^*a,b>$ by A4 then A6 and A1/A2.

(A9.) $h^*k=(h<k\pi,\pi'>)^*$

(A9. sol) $(h<k\pi,\pi'>)^*=(\epsilon<h^*\pi,\pi'><k\pi,\pi'>)^*=(\epsilon<h^*k\pi,\pi'>)^*=h^*k$ by A4, A6, and A5

(A10.) $\epsilon^*=1$

(A10. sol) $1=(\epsilon<1\pi,\pi'>)^*=(\epsilon)^*$ by A5, A7, and A5

3. Like we did in cartesian closed categories, let $a\times b$ denote $<a\pi,b\pi'>$ and $g^f=(g\epsilon<\pi,f\pi'>)^*$. Prove the following

(A11.) $(a\times b)(c\times d)=ac\times bd$

(A11 sol) $(a\times b)(c\times d)=<a\pi,b\pi'>(c\times d)=<a\pi(c\times d),b\pi'(c\times d)>=<ac\pi,bd\pi'>=ac\times bd$

(A12.) $g^f h=(g\epsilon<h\pi,f\pi'>)^*$

(A12. sol) $g^f h=(g\epsilon<\pi,f\pi'>)^*h=(g\epsilon<\pi,f\pi'><h\pi,\pi'>)^*=(g\epsilon<h\pi,f\pi'>)^*$

(A13.) $g^f k^h=(gk)^{(hf)}$

(A13. sol) $g^f k^h=(g\epsilon<k^h\pi,f\pi'>)^*=(g\epsilon<(k\epsilon<\pi\pi,h\pi'>)^*,f\pi'>)^*=(gk\epsilon<\pi\pi,h\pi'><1,f\pi'>)^*=(gk\epsilon<\pi,hf\pi'>)^*=(gk)^{(hf)}$

We can form a category of C-monoids where arrows are morphisms, called C-homomorphisms, that preserve the operations π, π', $(-)^*$, and $<-,->^*$. Given a monoid $\mathcal{M}$, we can also form the polynomial C-monoid $\mathcal{M}[x]$ by the usual construction of universal algebra (like we did for cartesian closed category): the elements of $\mathcal{M}[x]$ are polynomials (or words) built up from x and the elements of $\mathcal{M}$ using the C-monoid operations modulo the axioms A1–A5. In particular the mapping $h:\mathcal{M}\rightarrow\mathcal{M}[x]$ which sends every elements of $\mathcal{M}$ onto the corresponding constant polynomial in $\mathcal{M}[x]$ is a C-homomorphism.

C-monoids have also the property of functional completeness, that is for every polynomial $\varphi(x)$ in the indeterminate x, there exists a unique constant f in $\mathcal{M}$ such that $f<(x\pi')^*,1>=\varphi(x)$ in $\mathcal{M}[x]$.

Let $\rho_x\varphi(x)$ be defined inductively on the length of the word $\varphi(x)$ by

(i.) $\rho_x k=k\pi'$ if k is a constant

(ii.) $\rho_x x=\epsilon$

(iii.) $\rho_x<\psi(x),\xi(x)>=<\rho_x\psi(x),\rho_x\xi(x)>$

(iv.) $\rho_x(\xi(x)\psi(x))=\rho_x\xi(x)<\pi,\rho_x\psi(x)>$

(v.) $\rho_x(\psi(x)^*)=(\rho_x\psi(x)\alpha)^*$

where $\alpha=<\pi\pi,<\pi'\pi,\pi'>>$ (I am being a slightly informal with the equality here, since ideally, we would have used two different signs, one for identical words and one for the equality over equivalence classes; it won't matter much here, but it is good to keep it in mind).

4. As a consequence of functional completeness, prove that if $\varphi(x)$ is a polynomial in the indeterminate x over a C-monoid $\mathcal{M}$, then there exists a unique g in $\mathcal{M}$ such that $g\wr x=\varphi(x)$ where the binary operator '$\wr$' is as follows $g\wr a=\epsilon<g(a\pi')^*,1>$.

4. sol The natural candidate for g would be the transpose of f in $\mathcal{M}$, that is f^*, where the existence of f is ensured by functional completeness. By A8, one therefore gets $g\wr x=f^*\wr x=\epsilon<f^*(x\pi')^*,1>=f<(x\pi')^*,1>=\varphi(x)$.

5. Suppose one writes $\lambda_x\varphi(x)$ for $(\rho_x\varphi(x))^*$, where ρ_x is as above (using the λ-calculus notation). Rephrase (4.) using λ_x.

5. sol When $\varphi(x)$ is a constant k in $\mathscr{M}$, $f = \rho_x k = k\pi'$ which is in $\mathscr{M}$. Let's check. $(k\pi') < (x\pi')^*, 1 >= k = \varphi(x)$. When $\varphi(x)$ is a variable x, then $f = \rho_x x = \epsilon$ which is also in $\mathscr{M}$ for $\epsilon < (x\pi')^*, 1 >= (x\pi') < 1, 1 >= x1 = x = \varphi(x)$ (by A8). When $\varphi(x)$ has the form $< \psi(x), \xi(x) >$ and both $\psi(x)$ and $\xi(x)$ are of length 1 (either constants or variables), we set $f =< \rho_x\psi(x), \rho_x\xi(x) >$ which is also in $\mathscr{M}$ since we've just seen that ρ_x is acting as a binder for x. One gets $< \rho_x\psi(x), \rho_x\xi(x) >< (x\pi')^*, 1 >=< \rho_x\psi(x) < (x\pi')^*, 1 >, \rho_x\xi(x) < (x\pi')^*, 1 >>=< \psi(x), \xi(x) >= \varphi(x)$ as desired (the last equality holds because of the length of the operands and what we have just seen). Thus g is defined inductively as $f^* = (\rho_x\varphi(x))^*$. This further gives $\varphi(x) = g \wr x = (\lambda_x\varphi(x)) \wr x$ (morally, one is applying the functional abstraction of $\varphi(x)$, which is $\lambda_x\varphi(x)$ to x to get, well, $\varphi(x)$). Likewise for the other cases, and by induction on the length of $\varphi(x)$ one gets the desired f for any $\varphi(x)$ (constructively!).

6. Use the universal property of $\mathscr{M}[x]$ to state the β reduction. (The universal property of $\mathscr{M}[x]$ says that for every C-homomorphism $f : \mathscr{M} \to \mathscr{B}$ and every element $B \in \mathscr{B}$, there exists a unique C-homomorphosm $f_B : \mathscr{M}[x] \to \mathscr{B}$ such that $f_B h = f$ and $f_B(x) = B$.)

6. sol We want to show that, for any element a of $\mathscr{M}$, $(\lambda_x\varphi(x)) \wr a = \varphi(a)$. We just proved that $(\lambda_x\varphi(x)) \wr x = \varphi(x)$.

The universal property of h with the identity over $\mathscr{M}$ as f ensures the existence, for each element a of $\mathscr{M}$ of a C-homomorphism $(1_{\mathscr{M}})_a : \mathscr{M}[x] \to \mathscr{M}$ such that $(1_{\mathscr{M}})_a h = 1_{\mathscr{M}}$ and $(1_{\mathscr{M}})_a(x) = a$ for every $a \in \mathscr{M}$. In particular, $(1_{\mathscr{M}})_a\varphi(x) = \varphi(a)$ since $1_{\mathscr{M}}$ sends x to a and leaves the rest as is. Let $g = \lambda_x\varphi(x)$. We have $(1_{\mathscr{M}})_a(g \wr x) = (1_{\mathscr{M}})_a(\epsilon < g(x\pi')^*, 1 >) = \epsilon < g(a\pi')^*, 1 >= g \wr a$, and $(1_{\mathscr{M}})_a\varphi(x) = \varphi(a)$. Since $g \wr x = \varphi(x)$, we get $g \wr a = \varphi(a)$ for all $a \in \mathscr{M}$.

An interesting feature of C-monoids is that they enjoy the following fixed point theorem.

7. Prove that for every polynomial $\varphi(x)$ in $\mathscr{M}[x]$, there exists an element $a \in \mathscr{M}$ such that $\varphi(a) = a$.

7. sol Let $a = \lambda_x\varphi(x \wr x)$. Then $a \wr a = (\lambda_x\varphi(x \wr x)) \wr a = \varphi(a \wr a)$. What's the meaning of $\varphi(x \wr x)$ here?

8. Do you think that a C-monoid can incorporate propositional logic? Comment.

8. sol Propositional logic has a unary negation operation that doesn't have a fixed point which contradicts the result we just proved.

Bonus B. Show that in any cartesian closed poset with joins $p \vee q$, the following law of IPC (Intuitionistic Propositional Calculus) holds

$$((p \vee q) \Rightarrow r) \Rightarrow ((p \Rightarrow r) \wedge (q \Rightarrow r))$$

Generalize this result to an arbitrary category (not necessarily poset) by showing that there is always an arrow of the corresponding form.

Bonus BB. A functor $F : \mathscr{A} \to \mathscr{B}$ is *essentially surjective on objects* if for all $B \in \mathscr{B}$, there exists $A \in \mathscr{A}$ such that $F(A) \cong B$. Prove that a functor is an equivalence if and only if it is faithful, full, and essentially surjective on objects (restrict to small categories).