Exam 2022-2023

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We established a formal link between typed λ-calculi and cartesian closed categories (Curry-Howard-Lambek isomorphism). But what if we remove types? To which structure the pure (untyped) λ-calculus is isomorphic? Let's see!

Recall that a (small) category with one object is a monoid, that is a semigroup with a unitary element. (recall that small means that the class of morphisms between any two elements is a set. It is a technicality that you can drop in what follows.)

0. Describe a small closed cartesian category with only one object (elements and arrows are to be made explicit).

A C-monoid (for Curry or Cartesian) is a monoid $\mathcal{M}$ that is almost a cartesian closed category in the sense that it enjoys similar structure. Technically $\mathcal{M}$ has an extra structure $(\pi, \pi', \epsilon, *, <>)$, where π, π', and ϵ are elements (or nullary operations) of $\mathcal{M}$, $(-)^*$ is a unary operation and $< -, - >$ is a binary operation satisfying the following identities for all a, b, c, h, and k (where composition is denoted by simple concatenation, so ab denotes $a \circ b$):

(A1.) $\pi < a, b >= a$

(A2.) $\pi' < a, b >= b$

(A3.) $< \pi c, \pi' c >= c$

(A4.) $\epsilon < h^* \pi, \pi' >= h$

(A5.) $(\epsilon < k\pi, \pi' >)^* = k$

1. Recall briefly what do these axioms refer to in a standard cartesian closed category? (observe that the type subscript is omitted since there exists only one element.)

1.' What's missing to have a full cartesian closed category?

2. Using the above axioms, prove the following consequences (for all a, b, c, h, and k)

(A6.) $< a, b > c =< ac, bc >$

(A7.) $< \pi, \pi' >= 1$

(A8.) $\epsilon < h^* a, b >= h < a, b >$

(A9.) $h^* k = (h < k\pi, \pi' >)^*$

(A10.) $\epsilon^* = 1$

3. Like we did in cartesian closed categories, let $a \times b$ denote $< a\pi, b\pi' >$ and $g^f = (g\epsilon < \pi, f\pi' >)^*$. Prove the following

(A11.) $(a \times b)(c \times d) = ac \times bd$

(A12.) $g^f h = (g\epsilon < h\pi, f\pi' >)^*$

(A13.) $g^f k^h = (gk)^{(hf)}$

We can form a category of C-monoids where arrows are morphisms, called C-homomorphisms, that preserve the operations π, π', $(-)^*$, and $< -, - >^*$. Given a monoid $\mathscr{M}$, we can also form the polynomial C-monoid $\mathscr{M}[x]$ by the usual construction of universal algebra (like we did for cartesian closed category): the elements of $\mathscr{M}[x]$ are polynomials (or words) built up from x and the elements of $\mathscr{M}$ using the C-monoid operations modulo the axioms A1–A5. In particular the mapping $h : \mathscr{M} \to \mathscr{M}[x]$ which sends every elements of $\mathscr{M}$ onto the corresponding constant polynomial in $\mathscr{M}[x]$ is a C-homomorphism.

C-monoids have also the property of functional completeness, that is for every polynomial $\varphi(x)$ in the indeterminate x, there exists a unique constant f in $\mathscr{M}$ such that $f < (x\pi')^*, 1 >= \varphi(x)$ in $\mathscr{M}[x]$.

Let $\rho_x \varphi(x)$ be defined inductively on the length of the word $\varphi(x)$ by

(i.) $\rho_x k = k\pi'$ if k is a constant

(ii.) $\rho_x x = \epsilon$

(iii.) $\rho_x < \psi(x), \xi(x) >=< \rho_x \psi(x), \rho_x \xi(x) >$

(iv.) $\rho_x(\xi(x)\psi(x)) = \rho_x \xi(x) < \pi, \rho_x \psi(x) >$

(v.) $\rho_x(\psi(x)^*) = (\rho_x \psi(x)\alpha)^*$

where $\alpha =< \pi\pi, < \pi'\pi, \pi' >>$ (I am being a slight informal with the equalities here, since ideally, we would have used two different signs, one for identities over words and one equality over equivalence classes; it won't matter much here, but keep it in mind).

4. As a consequence of functional completness, prove that if $\varphi(x)$ is a polynomial in the indeterminate x over a C-monoid $\mathscr{M}$, then there exists a unique g in $\mathscr{M}$ such that $g \wr x = \varphi(x)$ where the binary operator '$\wr$' is as follows $g \wr a = \epsilon < g(a\pi')^*, 1 >$.

5. Suppose one writes $\lambda_x \varphi(x)$ for $(\rho_x \varphi(x))^*$, where ρ_x is as above (using the λ-calculus notation). Rephrase (4.) using λ_x.

6. Use the universal property of $\mathscr{M}[x]$ to state the β reduction. (The universal property of $\mathscr{M}[x]$ says that for every C-homomorphism $f : \mathscr{M} \to \mathscr{B}$ and every element $B \in \mathscr{B}$, there exists a unique C-homomorphosm $f_B : \mathscr{M}[x] \to B$ such that $f_B h = f$ and $f_B(x) = B$.)

An interesting feature of C-monoids is that they enjoy the following fixed point theorem.

7. Prove that for every polynomial $\varphi(x)$ in $\mathscr{M}[x]$, there exists an element $a \in \mathscr{M}$ such that $\varphi(a) = a$.

8. Do you think that a C-monoid can incorporate propositional logic? Comment.

Bonus B. Show that in any cartesian closed poset with joins $p \vee q$, the following law of IPC (Intuitionistic Propositional Calculus) holds

$$((p \vee q) \Rightarrow r) \Rightarrow ((p \Rightarrow r) \wedge (q \Rightarrow r))$$

Generalize this result to an arbitrary category (not necessarily poset) by showing that there is always an arrow of the corresponding form.

Bonus BB. A functor $F : \mathscr{A} \to \mathscr{B}$ is *essentially surjective on objects* if for all $B \in \mathscr{B}$, there exists $A \in \mathscr{A}$ such that $F(A) \cong B$. Prove that a functor is an equivalence if and only if it is faithful, full, and essentially surjective on objects (restrict to small categories).