Exam 2020-2021 (Ingredients and Recipes)

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Appetizer

1. What are the subcategories of a group? Which are full? (go easy, don't bite your tongue!)
2. Consistency implies that the set of all sets cannot be a set. Why then are we allowed to talk, shamelessly, about *the category* of all categories?

1 Recipe

1. Nothing says in the definition of a subcategory that the structure of a group must be preserved! Thus subgroups are not the only subcategories. *Submonoids*, however, check all the requirements of a subcategory since their arrows (by definition) are closed by composition. Here we have only one object, so for the submonoid to be full it has to account for all arrows and therefore the only possible non-trivial full subcategory if the entire group. Of course, the empty category is trivially a full subcategory of any category (there is no work to be done!).

2. This is perhaps more a philosophical question than a (pure) mathematical or computer science question. I didn't expect a precise answer for such an open (and rabbit-hole like question) anyway.

The first reaction you should have by now though is that one cannot formulate in a straightforward way Russel's paradox for categories for two main reasons. The first is that there is no *global* binary operator $\in$ in category theory. In a sense the relations are all *local*. You don't have enough power to talk upfront about arrows or objects far apart unless you dig up some structure that allows you to do so. The second reason is that category theory is typed: there are objects, arrows, functors, etc., and the stated relations must respect these types. This comes hard-encoded by the fact that each arrow has a fixed domain and codomain. Nothing is floating around freely. In other words, the theory doesn't bother studying objects in isolation for it considers that the meaning of something (object or arrow or whatever) depends on where that thing lives and what relations it has with its surroundings.

This being said, once we work out the list of properties that *sets* must have, it is comforting to check that the category of all small categories is not small itself, and the category of all locally small categories is neither small nor locally small. In particular, the category **Set** is not an object of itself, that is, it is not a set. Said differently, it is not because one can define and talk about the category of all categories that such a construction has the same nature (ore more precisely size) of its objects or arrows; it is often larger as a matter of fact.

We can give a better (more precise) answer, but for that we need to dive a bit deeper into *universes* and *topoi* ... Does this mean that category theory is free of paradoxes (consistent)? Absolutely not! No body knows. And even if it happens to be so, Gödel torpedo applies, it won't be provable within the theory itself, and one has to drop completeness.

Main course

Let $\mathscr{A}$ denote a locally small category (that is $\mathscr{A}(A, B)$ is a set for any A, B objects of $\mathscr{A}$). Fix an object A in $\mathscr{A}$ and let $H^A : \mathscr{A} \to \mathbf{Set}$ be defined as:

- for any object B, $H^A(B) = \mathscr{A}(A, B)$, and
- for any arrow $f : B \to B'$, $H^A(f) : \mathscr{A}(A, B) \to \mathscr{A}(A, B')$ sends $p : A \to B$ to $f \circ p : A \to B'$

1. Check that H^A is a functor.
2. When $\mathscr{A}$ is $\mathbf{Set}$, show that for any set B, $H^1(B) \cong B$ naturally in B (1 being the usual terminal object of $\mathbf{Set}$).

Notation: The functor H^A can be defined over any locally small category, there is nothing special about $\mathscr{A}$ except being locally small. For instance if $F : \mathscr{A} \to \mathscr{B}$ is a functor between two locally small categories, then $H^{F(A)}$ will denote the same construction except that now it is seen as an object of $[\mathscr{B}, \mathbf{Set}]$.

Let $\mathscr{A} \underset{G}{\overset{F}{\underset{\bot}{\rightleftarrows}}} \mathscr{B}$ be locally small categories (notice that F is left adjoint to G).

3. Show that $H^A \circ G \cong H^{F(A)}$ as objects of the functor category $[\mathscr{B}, \mathbf{Set}]$ (recall that arrows of a functor category are natural transformations).
4. Deduce that any set-valued functor $G : \mathscr{A} \to \mathbf{Set}$ with a left adjoint is isomorphic to H^A for some A in $\mathscr{A}$.

2 Recipe

Many identities used here for the pairing and product of arrows are detailed in the appendix "Drinks" at the very end. Check it out from now and then to hydrate yourself while enjoying your meal.

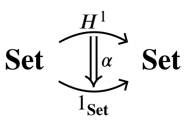
1. We need to check that $H^A(g \circ f) = H^A(g) \circ H^A(f)$ for any $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$, and that $H^A(1_B) = 1_{H^A(B)}$ for any B object of $\mathscr{A}$. In the category $\mathbf{Set}$, we know that functions are entirely determined by their action on the *elements* of their domain (that is maps with domain 1). Thus, by extensionality, we can show that the identity function over a set S is exactly the function satisfying $1_S \circ s = s$ for any $s : 1 \to S$. Given this fact, together with the identity laws of a category, one proves that $H^A(1_B)$ is (extensionally) equal to $1_{H^A(B)}$. Indeed for all p in $\mathscr{A}(A, B)$

$$H^A(1_B)(p) = 1_B \circ p = p = 1_{H^A(B)}(p).$$

To prove that H^A distributes over composition, we again use extensionality as well as the associativity axiom of the composition

$$H^A(g \circ f)(p) = (g \circ f) \circ p = g \circ (f \circ p) = H^A(g)(f \circ p) = H^A(g)(H^A(f)(p)) = (H^A(g) \circ H^A(f))(p).$$

2. We have two functors H^1 and $1_{\mathbf{Set}}$ and we need to make explicit a *natural isomorphism* α



Recall that *function sets* exist in **Set** for any two sets. They come equipped with an evaluation mapping satisfying a universality property (this is an axiom!). In particular, the function set $\mathscr{A}(1, B)$ comes together with a function $\varepsilon_B : \mathscr{A}(1, B) \times 1 \to B$ such that, for each X and $q : X \times 1 \to B$, there exists a unique function $\bar{q} : X \to \mathscr{A}(1, B)$ making the following diagram commute (1_1 denotes the identity on the terminal object 1)

$$\begin{array}{ccc} X \times 1 & & \\ {\scriptstyle \bar{q}\times 1_1}\downarrow & \searrow^{q} & \\ \mathscr{A}(1,B) \times 1 & \xrightarrow[\varepsilon_B]{} & B \end{array} \tag{1}$$

Recall that product on arrows is different from pairing. Indeed, the product of arrows is induced by the universality property of the product on objects:

$$\overline{q} \times 1_1 = \langle \overline{q} \circ \pi_{X,1}, 1_1 \circ \pi'_{X,1} \rangle = \langle \overline{q} \circ \pi_{X,1}, \pi'_{X,1} \rangle.$$

We instantiate diagram (1) with 1 as X and $b \circ \pi'_{1,1}$ as q where $b : 1 \to B$. By universality, there exists a *unique function* $b^* : 1 \to \mathscr{A}(1, B)$ making the following diagram commute

$$\begin{array}{ccc} 1 \times 1 & \xrightarrow{\pi'_{1,1}} & 1 \\ {\scriptstyle b^*\times 1_1}\downarrow & \searrow^{b\circ\pi'_{1,1}} & \downarrow{\scriptstyle b} \\ \mathscr{A}(1,B) \times 1 & \xrightarrow[\varepsilon_B]{} & B \end{array} \tag{2}$$

where we used the more standard notation b^* instead of $\overline{b \circ \pi'_{1,1}}$ (b^* is called the *name* of b by Lawvere). Since $1 \times 1 \xrightarrow{\pi'_{1,1}} 1$ is an isomorphism (the inverse arrow is $\langle 1_1, 1_1 \rangle$), it follows that b and b^* are in one-to-one correspondence: each element b of B has a corresponding function b^* in $\mathscr{A}(1, B)$, and

$$\varepsilon_B \circ \langle b^*, 1_1 \rangle = \varepsilon_B \circ (b^* \times 1_1) \circ \langle 1_1, 1_1 \rangle = b \circ \pi'_{1,1} \circ \langle 1_1, 1_1 \rangle = b \circ 1_1 = b$$

Let $\alpha_B : \mathscr{A}(1, B) \to B$ denote such an isomorphism. That is, for each $b^* : 1 \to \mathscr{A}(1, B)$

$$\alpha_B(b^*) = \alpha_B \circ b^* = \varepsilon_B \circ \langle b^*, 1_1 \rangle = b$$

It remains to show the naturality condition

$$\begin{array}{ccc} \mathscr{A}(1,B) & \xrightarrow{H^1(f)} & \mathscr{A}(1,B') \\ {\scriptstyle \alpha_B}\downarrow & & \downarrow{\scriptstyle \alpha_{B'}} \\ B & \xrightarrow[f]{} & B' \end{array}$$

We prove $\alpha_{B'} \circ H^1(f) = f \circ \alpha_B$ by showing that the equality holds for each element of $\mathscr{A}(1, B)$, that is for each map $b^* : 1 \to \mathscr{A}(1, B)$. Notice that, with respect to these definitions, $H^1(f)$ maps the name of b to the name of $f \circ b$, in other words, formally $H^1(f) \circ b^* = (f \circ b)^*$:

$$\begin{aligned} \alpha_{B'} \circ H^1(f) \circ b^* &= \alpha_{B'} \circ (f \circ b)^* \\ &= \varepsilon_{B'} \circ \langle (f \circ b)^*, 1_1 \rangle \\ &= f \circ b \\ &= f \circ \alpha_B \circ b^* \end{aligned}$$

By extensionality, $\alpha_{B'} \circ H^1(f) = f \circ \alpha_B$ which proves the naturality in B.

3. For each B in $\mathscr{B}$, we are looking for an isomorphism $(H^A \circ G)(B) \cong (H^{F(A)})(B)$ natural in B. By definition, $(H^A \circ G)(B) = \mathscr{A}(A, G(B))$ and $(H^{F(A)})(B) = \mathscr{B}(F(A), B)$. Since F is left adjoint to G, we are already provided such an isomorphism. It remains to check the naturality in B, that is that the following diagram commutes for some $q : B \to B'$ in $\mathscr{B}$

$$\begin{array}{ccc} \mathscr{A}(A, G(B)) & \xrightarrow{H^A(G(q))} & \mathscr{A}(A, G(B')) \\ \downarrow & & \downarrow \\ \mathscr{B}(F(A), B) & \xrightarrow{H^{F(A)}(q)} & \mathscr{B}(F(A), B') \end{array}$$

We prove this by extentionality (The diagram is in **Set**). Let f be an element of $\mathscr{A}(A, G(B))$. We want to show that

$$\overline{G(q) \circ f} = q \circ \overline{f}$$

which is provided by the very naturality axioms of adjoints.

4. To prove that $H^1 \circ G \cong G$ as objects in $[\mathscr{A}, \mathbf{Set}]$, we need to prove that,

$$(H^1 \circ G)(A) \cong G(A) \quad \text{naturally in } A$$

From (2.) we get the desired equivalence, namely

$$(H^1 \circ G)(A) = H^1(G(A)) \cong G(A).$$

From (3.) with $A = 1$, we get that $G \cong H^1 \circ G \cong H^{F(1)}$.

Desert

1. Prove that the identities defined on a Kleisli category are identities and that the composition as defined is indeed associative.

2. Detail the proofs of the two derived rules (or operators on proofs) of positive intuitionistic logic f^* and f_* (slide 16).

3. Prove that $(f_*)^* = (f^*)_* = f$ where $f : A \to B$ and A and B are formulas.

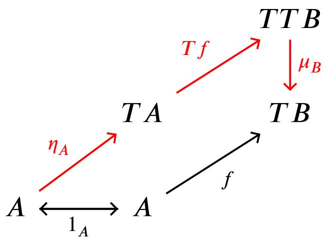
3 Recipe

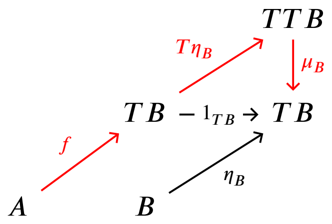
1. We need to check that for any arrow $f : A \twoheadrightarrow B$,

$$\eta_B \circ_K f = f = f \circ_K \eta_A$$

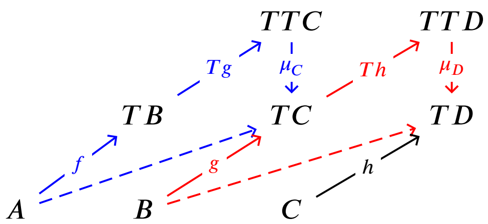
where $\eta_A : A \to TA$, $\eta_B : B \to TB$, and $\circ_K$ denotes the composition in Kleisli categories (to distinguish it from the original composition). The two equalities can be read from the following commuting diagrams

(both $\eta_B \circ_K f$ and $f \circ_K \eta_A$ are colored in red)

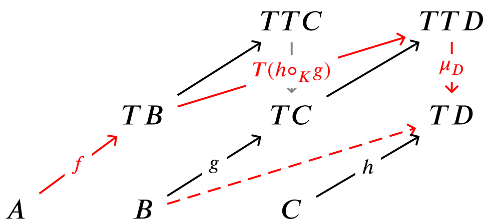


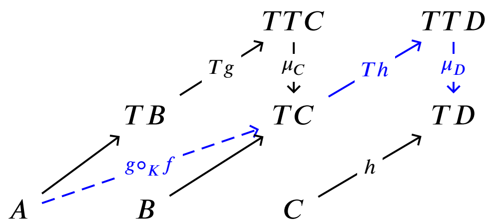


It remains to prove associativity. We want to show $(h \circ_K g) \circ_K f = h \circ_K (g \circ_K f)$. In the following diagram $h \circ_K g$ is dashed in red and $g \circ_K f$ is dashed in blue (the diagram commutes by definition of $\circ_K$):

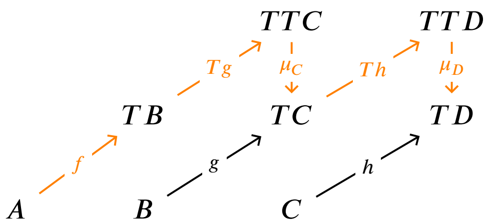


We next draw $(h \circ_K g) \circ_K f$ (left) and $h \circ_K (g \circ_K f)$ (right)





and both paths amount in fact to the "canonical" $h \circ_K g \circ_K f$ depicted below on the same diagram



2.

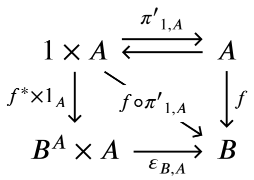
$$\frac{\dfrac{\mathbf{T} \wedge A \xrightarrow{\pi'_{\mathbf{T},A}} A \qquad A \xrightarrow{f} B}{\mathbf{T} \wedge A \xrightarrow{f \circ \pi'_{\mathbf{T},A}} B}}{\mathbf{T} \xrightarrow{\overline{f \circ \pi'_{\mathbf{T},A}}} B \Leftarrow A}$$

$$\frac{\dfrac{\dfrac{A \xrightarrow{A\bullet} \mathbf{T} \qquad \mathbf{T} \xrightarrow{g} B \Leftarrow A}{A \xrightarrow{g \circ A\bullet} B \Leftarrow A} \qquad A \xrightarrow{1_A} A}{A \xrightarrow{\langle g \circ A\bullet, 1_A \rangle} (B \Leftarrow A) \wedge A} \qquad (B \Leftarrow A) \wedge A \xrightarrow{\varepsilon_{B,A}} B}{A \xrightarrow{\varepsilon_{B,A} \circ \langle g \circ A\bullet, 1_A \rangle} B}$$

3. Let $f : A \to B$ and $g : 1 \to B^A$ be arrows and objects in a cartesian closed category ('1' being the product of zero elements or the terminal object of the category). We have

$$
\begin{aligned}
(f^*)_* &= \varepsilon_{B,A} \circ \langle f^* \circ A\bullet, 1_A \rangle \circ 1_A \\
&= \varepsilon_{B,A} \circ \langle f^* \circ A\bullet, 1_A \rangle \circ \pi'_{1,A} \circ \langle A\bullet, 1_A \rangle \\
&= \varepsilon_{B,A} \circ \langle f^* \circ \pi_{1,A}, \pi'_{1,A} \rangle \circ \langle A\bullet, 1_A \rangle \\
&= \varepsilon_{B,A} \circ (f^* \times 1_A) \circ \langle A\bullet, 1_A \rangle \\
&= f
\end{aligned}
$$

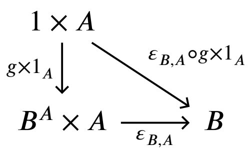
where the last equation can be seen diagramatically (notice that $\pi'_{1,A}$ is an isomorphism and its inverse is $\langle A\bullet, 1_A \rangle$).



This diagram is in fact the general case of diagram (2). For the remaining equation, we also have

$$
\begin{aligned}
(g_*)^* &= \overline{g_* \circ \pi'_{1,A}} \\
&= \overline{\varepsilon_{B,A} \circ \langle g \circ A\bullet, 1_A \rangle \circ \pi'_{1,A}} \\
&= \overline{\varepsilon_{B,A} \circ \langle g \circ \pi_{1,A}, \pi'_{1,A} \rangle} \\
&= \overline{\varepsilon_{B,A} \circ g \times 1_A} \\
&= g
\end{aligned}
$$

The last equation comes from the uniqueness of the arrow $1 \to B^A$ that makes the following diagram commute

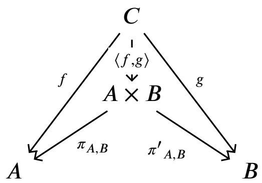


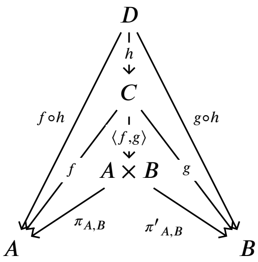
A Drinks

A.1 Universality of the product

The universality property of the product is much richer than it appears at the first sight. Below we state some important identities. For any arrows and objects $C \xrightarrow{f} A$, $C \xrightarrow{g} B$, and $D \xrightarrow{h} C$ in a cartesian category, denoting the unique map $C \to A \times B$ by $\langle f, g \rangle$, one has:

1. $\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle$,
2. $\pi_{A,B} \circ \langle f, g \rangle = f$ and $\pi'_{A,B} \circ \langle f, g \rangle = g$
3. $\langle \pi_{A,B}, \pi'_{A,B} \rangle = 1_{A \times B}$





A.2 Some isomorphisms

In any cartesian category $\mathscr{A}$, one has $A \times 1 \cong A$ for any object A of $\mathscr{A}$. The isomorphism is as follows

$$A \times 1 \underset{\langle 1_A, A\bullet \rangle}{\overset{\pi_{A,1}}{\rightleftarrows}} A \qquad \text{where} \quad A \xrightarrow{A\bullet} 1 \quad \text{(1 is terminal, so } A\bullet \text{ exists and is unique).}$$

According to the identities above, one has

$$\pi_{A,1} \circ \langle 1_A, A\bullet \rangle = 1_A.$$

Since 1 is terminal, one also has

$$A \times 1 \xrightarrow{(A \times 1)\bullet = \pi'_{A,1} = (A\bullet) \circ \pi_{A,1}} 1.$$

Thus (the first and last equalities comes from the first and last identities above)

$$\langle 1_A, A\bullet \rangle \circ \pi_{A,1} = \langle 1_A \circ \pi_{A,1}, (A\bullet) \circ \pi_{A,1} \rangle = \langle \pi_{A,1}, \pi'_{A,1} \rangle = 1_{A \times 1}.$$