Exam 2020-2021

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Appetizer

1. What are the subcategories of a group? Which are full? (go easy, don't bite your tongue!)
2. Consistency implies that the set of all sets cannot be a set. Why then are we allowed to talk, shamelessly, about *the category* of all categories?

Main course

Let $\mathscr{A}$ denote a locally small category (that is $\mathscr{A}(A, B)$ is a set for any A, B objects of $\mathscr{A}$). Fix an object A in $\mathscr{A}$ and let $H^A : \mathscr{A} \to \mathbf{Set}$ be define as:

- for any object B, $H^A(B) = \mathscr{A}(A, B)$, and
- for any arrow $f : B \to B'$, $H^A(f) : \mathscr{A}(A, B) \to \mathscr{A}(A, B')$ sends $p : A \to B$ to $f \circ p : A \to B'$

1. Check that H^A is a functor.
2. When $\mathscr{A}$ is **Set**, show that for any set B, $H^1(B) \cong B$ naturally in B (1 being the usual terminal object of **Set**).

Notation: The functor H^A can be defined over any locally small category, there is nothing special about $\mathscr{A}$ except being locally small. For instance if $F : \mathscr{A} \to \mathscr{B}$ is a functor between two locally small categories, then $H^{F(A)}$ will denote the same construction except that now it is seen as an object of $[\mathscr{B}, \mathbf{Set}]$.

Let $\mathscr{A} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathscr{B}$ be locally small categories (notice that F is left adjoint to G).

3. Show that $H^A \circ G \cong H^{F(A)}$ as objects of the functor category $[\mathscr{B}, \mathbf{Set}]$ (recall that arrows of a functor category are natural transformations).
4. Deduce that any set-valued functor $G : \mathscr{A} \to \mathbf{Set}$ with a left adjoint is isomorphic to H^A for some A in $\mathscr{A}$.

Desert

1. Prove that the identities defined on a Kleisli category are identities and that the composition as defined is indeed associative.
2. Detail the proofs of the two derived rules (or operators on proofs) of positive intuitionistic logic f^* and f_* (slide 16).
3. Prove that $(f_*)^* = (f^*)_* = f$ where $f : A \to B$ and A and B are formulas.