TD4

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Exercise 1 Let $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ be categories and functors. Suppose $F \dashv G$ with unit η and counit ε, prove that $(T = GF, \eta, \mu = G\epsilon F)$ determines a monad on $\mathscr{A}$.

Exercise 2 Show that for any three objects A, B, C in a cartesian closed category, $(A \times B)^C \cong A^C \times B^C$ and $(A^B)^C \cong A^{B \times C}$.

Exercise 3 Let $\mathscr{A}$ be a CCC. Fix an object C. Consider the functors

$$\begin{aligned}(- \times C) : \mathscr{A} &\to \mathscr{A} \\ A &\mapsto A \times C\end{aligned} \qquad\qquad \begin{aligned}(-)^C : \mathscr{A} &\to \mathscr{A} \\ A &\mapsto A^C\end{aligned}$$

Show that $(- \times C)$ is left adjoint to $(-)^C$.

Exercise 4 Consider the category of pointed sets, that is sets equipped with a special element $(A, a \in A)$ with maps $f : (A, a) \to (B, b)$ being those functions $f : A \to B$ such that $f(a) = b$. Is such category cartesian closed?

Exercise 5 (bonus) Show that in any cartesian closed poset with joins $p \vee q$, the following law of IPC (Intuitionistic Propositional Calculus) holds

$$((p \vee q) \Rightarrow r) \Rightarrow ((p \Rightarrow r) \wedge (q \Rightarrow r))$$

Generalize this result to an arbitrary category (not necessarily poset) by showing that there is always an arrow of the corresponding form.