

TD3

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Adjoints

Recall that an object I in a category \mathcal{A} is *initial* if for every $A \in \mathcal{A}$, there exists a unique map $I \rightarrow A$. An object $T \in \mathcal{A}$ is *terminal* if for every $A \in \mathcal{A}$, there exists a unique maps $A \rightarrow T$. The terminal object of **CAT** is the category $\mathbf{1}$ (with one object and the identity on it).

Exercise 0 Initial and terminal objects can be described as adjoints. Think about this and try to make it explicit.

Exercise 1 What can be said about adjunctions of discrete categories?

Exercise 2 Show that left adjoints preserve initial objects: that is, if $\mathcal{A} \xrightleftharpoons[G]{F} \mathcal{B}$ are categories and functors with $F \dashv G$, and I is an initial object of \mathcal{A} , then $F(I)$ is an initial object of \mathcal{B} . Dually, show that right adjoints preserve terminal objects.

Exercise 3 Given an adjunction $F \dashv G$ with unit η and counit ε , prove that the following triangles commute.

$$\begin{array}{ccc}
 F & \xrightarrow{F\eta} & FGF \\
 & \searrow 1_F & \downarrow \varepsilon F \\
 & & F
 \end{array}
 \qquad
 \begin{array}{ccc}
 G & \xrightarrow{\eta G} & GFG \\
 & \searrow 1_G & \downarrow G\varepsilon \\
 & & G
 \end{array}$$

Exercise 4 Let $A \xrightleftharpoons[g]{f} B$ be order-preserving maps between ordered sets. Prove that the following conditions are equivalent:

- (a) for all $a \in A$ and $b \in B$, $f(a) \leq b \iff a \leq g(b)$
- (b) $a \leq g(f(a))$ for all $a \in A$ and $f(g(b)) \leq b$ for all $b \in B$.

Exercise 5 (bonus) Given a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ and a category \mathcal{L} , there is a functor

$$F^* : [\mathcal{B}, \mathcal{L}] \rightarrow [\mathcal{A}, \mathcal{L}]$$

defined on objects $Y \in [\mathcal{B}, \mathcal{L}]$ by $F^*(Y) = Y \circ F$ and on maps α by $F^*(\alpha) = \alpha F$. Show that any adjunction $F \dashv G$ with $\mathcal{A} \xrightleftharpoons[G]{F} \mathcal{B}$ and category \mathcal{L} give rise to an adjunction $G^* \dashv F^*$ with

$$[\mathcal{A}, \mathcal{L}] \xrightleftharpoons[F^*]{G^*} [\mathcal{B}, \mathcal{L}] \quad .$$