TD2

Khalil Ghorbal

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Natural transformations

Exercise 1 Let 2 be the discrete category with two objects. Prove that the functor category $[2, \mathscr{B}]$ is isomorphic to the product category $\mathscr{B} \times \mathscr{B}$. (The notation $\mathscr{B}^2$ for $[2, \mathscr{B}]$ is now justified.)

Exercise 2 A permutation of a set X is a bijection $X \to X$. Let $\mathbf{Sym}(X)$ denote the set of permutations of X. A total order on a set X is an order $\leq$ such that for all $x, y \in X$, either $x \leq y$ or $y \leq x$. So a total order amounts to a way to placing the elements of X in a sequence. Let $\mathbf{Ord}(X)$ denote the set of total orders of X. Let $\mathscr{A}$ denote the category of finite sets and bijections.

(a) There is a canonical way to regard both **Sym** and **Ord** as functors from $\mathscr{A}$ to the category of sets **Set**. Make it explicit.

(b) Show that there is no natural transformation $\mathbf{Sym} \to \mathbf{Ord}$.

(c) For an n-element set X, how many elements do the sets $\mathbf{Sym}(X)$ and $\mathbf{Ord}(X)$ have?

Conclude that $\mathbf{Sym}(X) \cong \mathbf{Ord}(X)$ but no naturally for all $X \in \mathscr{A}$. (This is an example where maps in $\mathscr{A}$ are isomorphic in the standard sense with no natural way to match them up.)

Exercise 3 A functor $F : \mathscr{A} \to \mathscr{B}$ is *essentially surjective on objects* if for all $B \in \mathscr{B}$, there exists $A \in \mathscr{A}$ such that $F(A) \cong B$. Prove that a functor is an equivalence if and only if it is faithful, full, and essentially surjective on objects.

Exercise 4 (bonus) Show that equivalence of categories is an equivalence relation.

Adjoints

Recall that an object I in a category $\mathscr{A}$ is *initial* if for every $A \in \mathscr{A}$, there exists a unique map $I \to A$. An object $T \in \mathscr{A}$ is *terminal* if for every $A \in \mathscr{A}$, there exists a unique maps $A \to T$. The terminal object of **CAT** is the category 1 (with one object and the identity on it).

Exercise 1 Initial and terminal objects can be described as adjoints. Think about this and try to make it explicit.

Exercise 2 Show that left adjoints preserve initial objects: that is, if $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ are categories and functors with $F \dashv G$, and I is an initial object of $\mathscr{A}$, then $F(I)$ is an initial object of $\mathscr{B}$. Dually, show that right adjoints preserve terminal objects.

Exercise 3 Given an adjunction $F \dashv G$ with unit η and counit ε, prove that the following triangles commute.

$$
\begin{array}{ccc}
F & \xrightarrow{F\eta} & FGF \\
& {\scriptstyle 1_F}\searrow & \downarrow{\scriptstyle \varepsilon F} \\
& & F
\end{array}
\qquad\qquad
\begin{array}{ccc}
G & \xrightarrow{\eta G} & GFG \\
& {\scriptstyle 1_G}\searrow & \downarrow{\scriptstyle G\varepsilon} \\
& & G
\end{array}
$$

Exercise 4 Let $A \underset{g}{\overset{f}{\rightleftarrows}} B$ be order-preserving maps between ordered sets. Prove that the following conditions are equivalent:

(a) for all $a \in A$ and $b \in B$, $f(a) \leq b \iff a \leq g(b)$

(b) $a \leq g(f(a))$ for all $a \in A$ and $f(g(b)) \leq b$ for all $b \in B$.

Exercise 5 (bonus) Given a functor $F : \mathscr{A} \to \mathscr{B}$ and a category $\mathscr{L}$, there is a functor

$$F^* : [\mathscr{B}, \mathscr{L}] \to [\mathscr{A}, \mathscr{L}]$$

defined on objects $Y \in [\mathscr{B}, \mathscr{L}]$ by $F^*(Y) = Y \circ F$ and on maps α by $F^*(\alpha) = \alpha F$. Show that any adjunction $F \dashv G$ with $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ and category $\mathscr{L}$ give rise to an adjunction $G^* \dashv F^*$ with

$$[\mathscr{A}, \mathscr{L}] \underset{F^*}{\overset{G^*}{\rightleftarrows}} [\mathscr{B}, \mathscr{L}] \quad .$$