TD2

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Natural transformations

Exercise 0 Let 2 be the discrete category with two objects. Prove that the functor category $[2, \mathscr{B}]$ is isomorphic to the product category $\mathscr{B} \times \mathscr{B}$. (The notation $\mathscr{B}^2$ for $[2, \mathscr{B}]$ is now justified.)

Exercise 1 Let $\mathscr{A}$ and $\mathscr{B}$ be categories. Prove that $[\mathscr{A}, \mathscr{B}]^{\text{op}} \cong [\mathscr{A}^{\text{op}}, \mathscr{B}^{\text{op}}]$.

Exercise 2 Let $\mathscr{A} \underset{G}{\overset{F}{\Downarrow \alpha}} \mathscr{B}$ be a natural transformation. Prove that α is a natural isomorphism if and only if its components α_A are isomorphisms (in $\mathscr{B}$) for each $A \in \mathscr{A}$. (This can be taken as an equivalent definition to natural isomorphism.)

Exercise 3 A permutation of a set X is a bijection $X \to X$. Let $\mathbf{Sym}(X)$ denote the set of permutations of X. A total order on a set X is an order $\leq$ such that for all $x, y \in X$, either $x \leq y$ or $y \leq x$. So a total order amounts to a way to placing the elements of X in a sequence. Let $\mathbf{Ord}(X)$ denote the set of total orders of X. Let $\mathscr{A}$ denote the category of finite sets and bijections.

(a) There is a canonical way to regard both **Sym** and **Ord** as functors from $\mathscr{A}$ to the category of sets **Set**. Make it explicit.

(b) Show that there is no natural transformation $\mathbf{Sym} \to \mathbf{Ord}$.

(c) For an n-element set X, how many elements do the sets $\mathbf{Sym}(X)$ and $\mathbf{Ord}(X)$ have?

Conclude that $\mathbf{Sym}(X) \cong \mathbf{Ord}(X)$ but no naturally for all $X \in \mathscr{A}$. (This is an example where maps in $\mathscr{A}$ are isomorphic in the standard sense with no natural way to match them up.)

Exercise 4 A functor $F : \mathscr{A} \to \mathscr{B}$ is *essentially surjective on objects* if for all $B \in \mathscr{B}$, there exists $A \in \mathscr{A}$ such that $F(A) \cong B$. Prove that a functor is an equivalence if and only if it is faithful, full, and essentially surjective on objects.

Exercise 5 (bonus) Show that equivalence of categories is an equivalence relation.