Exo 2 (TD4) $(A\times B)^C \cong A^C \times B^C$

$$\begin{tikzcd} I\times C \arrow[d, "\exists!\, \bar q\times 1_C"'] \arrow[dr, "\forall q"] & \\ (A\times B)^C\times C \arrow[r, "\varepsilon"'] & A\times B \end{tikzcd} \qquad A^C\times C \xrightarrow{\varepsilon_A} A \qquad B^C\times C \xrightarrow{\varepsilon_B} B$$

- $A^C\times B^C \longrightarrow (A\times B)^C$

$I = A^C\times B^C$

$q = (\varepsilon_A \circ (\pi_{A^C}, \pi_C), \varepsilon_B\circ(\pi_{B^C}, \pi_C))$

$: A^C\times B^C\times C \longrightarrow A\times B$

$\exists!\ \bar q : A^C\times B^C \longrightarrow (A\times B)^C,\ q = \varepsilon\circ(\bar q, 1_C)$

- $(A\times B)^C \longrightarrow A^C\times B^C$

$I = (A\times B)^C \qquad f : (A\times B)^C\times C \longrightarrow A$, $g : \ldots \longrightarrow B$

$f = \pi_A\circ\varepsilon \qquad \exists!\ \bar f, \bar g \qquad f = \varepsilon_A\circ(\bar f, 1_C)$

$g = \pi_B\circ\varepsilon \qquad g = \varepsilon_B\circ(\bar g, 1_C)$

$\bar f\circ\bar q : A^C\times B^C \longrightarrow A^C \qquad (\bar f, \bar g)\circ\bar q = 1_{A^C\times B^C}$

$h = \varepsilon_A\circ(\pi_{A^C}, \pi_C) = \varepsilon_A\circ(\bar h, 1_C)$, $\bar h$ unique donc $= \grave{a}\ \pi_{A^C}$

Or $\varepsilon_A\circ(\bar f\circ\bar q, 1_C) = \varepsilon_A\circ(\bar f, 1_C)\circ(\bar q, 1_C)$

$= \pi_A\circ\varepsilon\circ(\bar q, 1_C)$

$= \pi_A\circ q = \quad \circ(\pi_{A^C}, \pi_C)$

donc $\bar f\circ\bar q = \pi_{A^C}$

De même $\bar g\circ\bar q = \pi_{B^C}$ et $(\bar f, \bar g)\circ\bar q = 1_{A^C\times B^C}$

Cartesian category:

$A\times 1 \cong A$

$A\times B \cong B\times A$

$(A\times B)\times C \cong A\times(B\times C)$

cartesian closed category:

$A^1 \cong A$

$1^A \cong 1$

$(A\times B)^C \cong A^C\times B^C$

$A^{B\times C} \cong (A^C)^B \left(\cong (A^B)^C\right)$

bicartesian closed category

coproduit $+$

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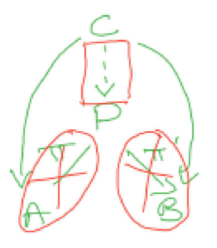
cartesian closed category $\mathcal{A}$

$\mathcal{A}(A\times B, C) \cong \mathcal{A}(A, C^B)$

$A = 1$ functional completeness

$\boxed{B \sim \mathcal{A}(1, C^B)}$



$$\begin{tikzcd} & 1\times B \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ 1 & & B \end{tikzcd}$$

$B \xrightarrow{f} C \quad \rightsquigarrow \quad 1 \xrightarrow{f\circ\pi'} C^B$

$1 \xrightarrow{g} C^B \quad \rightsquigarrow \quad B \xrightarrow{\varepsilon_C\circ(g\circ t_B, 1_B)} C$ with $B \xrightarrow{t_B} 1$

$$\begin{tikzcd} & B \arrow[dl, "t_B"'] \arrow[d] \arrow[ddr, "id_B"] & \\ 1 \arrow[d, "g"'] & C^B\times B \arrow[dl] \arrow[d, "\varepsilon_C"] \arrow[dr] & \\ C^B & C & B \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "FG"] \arrow[r, bend right, "1_{\mathcal{A}}"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \mathcal{A} \end{tikzcd}$$

$\mathcal{L}$ $x : 1 \longrightarrow A$ $\mathcal{L}[x]$

Proposition

For every polynomial $\varphi(x) : B \longrightarrow C$ in an indeterminate $x : 1 \longrightarrow A$ over a cartesian closed category $\mathcal{L}$ there is a unique arrow $f : A\times B \longrightarrow C$ in $\mathcal{L}$ such that

$$f\circ(x\circ t_B, 1_B) = \varphi(x)$$

Corollary: For every polynomial $\varphi(x) : 1 \longrightarrow C$ in an indeterminate $x : 1 \longrightarrow A$ over a cartesian closed category $\mathcal{L}$, there is a unique arrow $g : A \longrightarrow C$ in $\mathcal{L}$ such that $g\circ x = \varphi(x)$, and there is a unique arrow $h : 1 \longrightarrow C^A$ such that $\varepsilon_C\circ(h, x) = \varphi(x)$.

we denote h by $\lambda_{x:A}\varphi(x)$

$$h\,{}'\,x \triangleq \varepsilon_C\circ(h, x) = \varphi(x)$$

Def: The internal language $\mathbf{L}(\mathcal{A})$ of a cartesian closed category $\mathcal{A}$ with weak natural numbers object is defined as follows:

- its types are the objects of $\mathcal{A}$ with 1, $\underline{N}$, $A\times B$, B^A ...
- Terms of type / shape C are those polynomial expressions $\varphi(x) : 1 \longrightarrow C$ in the indeterminate $x : 1 \longrightarrow A$ which are obtained from x and the forming operators:

$$\frac{a : 1 \longrightarrow A \qquad b : 1 \longrightarrow B}{(a, b) : 1 \longrightarrow A\times B}$$

$$\frac{a : 1 \longrightarrow A \qquad f : A \longrightarrow B}{fa : 1 \longrightarrow B}$$

$$\frac{\varphi(x) : 1 \longrightarrow C}{\underbrace{\lambda_{x:A}\varphi(x)}_{h} : 1 \longrightarrow C^A}$$

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "a"'] & N \arrow[r, "S"] \arrow[d, dashed, "r"] & N \arrow[d, dashed, "r"] \\ & X \arrow[r, "f"] & X \end{tikzcd}$$

$r\circ 0 = a$, $r\circ S = f\circ r$

$M ::= x \mid M\,{}'\,N \mid \lambda x.\,M$

(β) $(\lambda x\, \varphi(x))\,{}'\,a = \varphi(a)$, whenever no free occurence in a becomes bound in $\varphi(a)$

(η) $\lambda x.(f\,{}'\,x) = f$ whenever f is independent from x

$f : B^A \qquad a : A$

$f\,{}'\,a : B$

$x : A \qquad f\,{}'\,x : B$

$\lambda x.\,\varphi(x) : B^A$

$\underline{0} \equiv \lambda x\, I \equiv \lambda x(\lambda x.\,x)$

$\underline{1} \equiv \lambda x\, x$

$\underline{1}\,{}'\,f = f$

$\underline{2}\,{}'\,f = f\circ f$

$I : A^A \qquad 1 : A^A$

$0 : (A^A)^A$

$A^A \cong (A^A)^A$

- Cart_N :
  - obj : cartesian closed cat with weak natural numbers obj
  - "→" : functors that respect the cartesian closed structure.
- λ-Calc :
  - obj : typed λ-calculi
  - "→" : translations $\phi : \mathcal{L} \longrightarrow \mathcal{L}'$ — if a of type A in $\mathcal{L}$ then $\phi(a)$ is of type $\phi(A)$ in $\mathcal{L}'$

Rq: we can have constructions satisfying $A \cong A\times A \cong A^A$ $A \neq 1$

Dana Scott 1972

Rq2: Coq : N, Ω Prop $\underline{2}^A$

Howard 1969

typed terms (constructions)

1980 Curry hommage

1982 Huet Coquand

CoC

Coq   Co-q

$\mathbf{L} : \mathrm{Cart}_N \longrightarrow \lambda\text{-Calc}$

is a functor ...

with an inverse, thus

$$\boxed{\mathrm{Cart}_N \cong \lambda\text{-Calc}}$$

obj	formulas/proposition	types
→	proofs	terms

Denotational Semantics

ational Semantics

- big step
- small step

ASM

(Alan)