Logic | Types / Lang
prop | λ-calcul
intui / classic
modal

---

(1) $Hom(A, 1) \cong \{*\}$
(2) $Hom(C, A \times B) \cong Hom(C, A) \times Hom(C, B)$
(3) $Hom(A, C^B) \cong Hom(A \times B, C)$

all entities are functions
all functions can be applied to objects

Schönfinkel Algebra (1924)

$A = (ob(A), ', I, K, S)$

- $I'a = a$ $\quad (I = fct(K, S))$
- $(K'a)'b = a$
- $((S'f)'g)'c = (f'c)'(g'c)$

Curry (~1930)

$x \in ob(A)$

every "polynomial" $\varphi(x)$ can be written as $f'x$ where $f \in ob(A)$ | $R[X]$ $a_0 + a_1X + a_2X^2 + \dots + a_nX^n$

(1941) A. Church $\quad f$ as $\lambda_x \varphi(x)$ $\quad x \longmapsto \varphi(x)$

(β) $(\lambda_x \varphi(x))'a = \varphi(a)$

(η) $\lambda_x (f'x) = f$

$I \equiv \lambda_x x$
$K \equiv \lambda_x \lambda_y x$
$S \equiv \lambda_u \lambda_v \lambda_z ((u'z)'(v'z))$

$I_A \quad A \Rightarrow A$
$K_{A,B} \quad A \Rightarrow (B \Rightarrow A)$
$S_{A,B,C} \quad (C \Rightarrow (B \Rightarrow A)) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow A))$
} axioms intuitionistic implicational logic

$f \circ g \equiv \lambda_x (f'(g'x)) \quad (f, g \in ob(A))$

$2'f \equiv f \circ f$
$1'f \equiv f$

$0 \equiv \lambda_x I$, $1 \equiv \lambda_x x$, $2 \equiv \lambda_x (x \circ x) \dots$

Church's numerals

$S'n \equiv \lambda_x (x \circ (n'x))$ (successor)
$m + n \equiv \lambda_x ((m'x) \circ (n'x))$
$mn \equiv m \circ n$
$m^n = n'm$

---

def: a deductive system is a graph with specified arrows

R1.a $\quad A \xrightarrow{1_A} A$

R1.b $\quad \dfrac{A \xrightarrow{f} B \quad B \xrightarrow{g} C}{A \xrightarrow{gf} C}$

objects (vertex) are called formulas
arrows are called proofs

$A \xrightarrow{f} B$

A conjunction calculus is a deductive system with:

- a formula T (= true)
- a binary operator $\wedge$ (= and) forming conjunctions $A \wedge B$

satisfying

R2. $\quad A \xrightarrow{O_A} T$
R3a $\quad A \wedge B \xrightarrow{\pi_{A,B}} A$
R3b $\quad A \wedge B \xrightarrow{\pi'_{A,B}} B$
R3c $\quad \dfrac{C \xrightarrow{f} A \quad C \xrightarrow{g} B}{C \xrightarrow{(f,g)} A \wedge B}$

for all formulas A, B, C and arrows f, g.

$$\dfrac{(R3b)\ A \wedge B \xrightarrow{\pi'_{A,B}} B \qquad A \wedge B \xrightarrow{\pi_{A,B}} A\ (R3a)}{A \wedge B \xrightarrow{(\pi'_{A,B}, \pi_{A,B})} B \wedge A}\ (R3c)$$

$$\dfrac{\dfrac{A \wedge C \xrightarrow{\pi_{A,C}} A \quad A \xrightarrow{f} B}{A \wedge C \longrightarrow B} \qquad \dfrac{A \wedge C \xrightarrow{\pi'_{A,C}} C \quad C \xrightarrow{g} D}{A \wedge C \longrightarrow D}}{A \wedge C \xrightarrow{f \wedge g} B \wedge D}$$

$f \wedge g \equiv (f \pi_{A,C}, g \pi'_{A,C})$

$$\dfrac{A \xrightarrow{f} B \qquad C \xrightarrow{g} D}{A \wedge C \xrightarrow{f \wedge g} B \wedge D}$$

A positive intuitionistic propositional calculus is a conjunction calculus with an additional binary operator $\Rightarrow$ (= if) $\Leftarrow$

R4a $\quad (A \Leftarrow B) \wedge B \xrightarrow{\varepsilon_{A,B}} A$

R4b $\quad \dfrac{C \wedge B \xrightarrow{h} A}{C \xrightarrow{h^*} (A \Leftarrow B)}$ $\quad (Hom(C \wedge B, A) \cong Hom(C, A^B))$

R4a' $\quad C \xrightarrow{\eta_{C,B}} (C \wedge B) \Leftarrow B$

R4b' $\quad \dfrac{D \xrightarrow{g} A}{(D \Leftarrow B) \xrightarrow{g \Leftarrow 1_B} (A \Leftarrow B)}$

(Deduction theorem
if $A \wedge B \vdash C$ then $A \vdash C \Leftarrow B$)

$\eta_{C,B} \equiv 1^*_{C \wedge B} \qquad g \Leftarrow 1_B \equiv (g\, \varepsilon_{D,B})^*$

$$\dfrac{A \xrightarrow{f} B}{T \xrightarrow{\bar{f}} (B \Leftarrow A)} \qquad \dfrac{T \xrightarrow{g} (B \Leftarrow A)}{A \xrightarrow{g'} B}$$

(CCC
$Hom(A, B) \cong Hom(1, B^A)$)

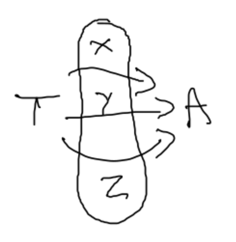
$\bar{f} \equiv (f \pi'_{1,A})^*$, $\quad g' \equiv \varepsilon_{B,A} (g\, O_A, 1_A)$

$T \xrightarrow{x} A \qquad \varphi(x): B \longrightarrow C$

$\mathcal{L} \qquad \mathcal{L}(x) \leftarrow$

$\mathbb{R} \qquad R[X] \qquad ax$

Deduction theorem:
with every proof $\varphi(x): B \longrightarrow C$ from the assumption $x: T \longrightarrow A$ there is an associated proof $f: A \wedge B \longrightarrow C$ in $\mathcal{L}$ not dependent on x.



$T \xrightarrow{x} A$
$T \xrightarrow{y} B$
$\mathcal{L}[x, y]$

$f \equiv \lambda_x \varphi(x) \qquad \forall_x \varphi(x) \qquad K_x \varphi(x)$